On Fragments of Analysis with Strengths of Finitely Iterated Inductive Definitions

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Inductive Types and System F

- Typical data structures:
  - natural numbers, lists, trees, ...

! all *inductive* types

- uniform framework: *polymorphism* (system F)
  - $\forall\alpha.(\alpha \to \alpha) \to \alpha \to \alpha$
  - $\forall\alpha.(\alpha \times \tau) \to \alpha \to \alpha$
  - $\forall\alpha.((\tau \to \alpha) \to \alpha) \to \alpha \to \alpha$, ...

$\triangleright$ but system F is far stronger (full second order arithmetic)!!

$\rightsquigarrow$ analyse the fragment actually needed

$$\boxed{\mu X\mathcal{F}(X) = \forall\alpha.(\mathcal{F}(\alpha) \to \alpha) \to \alpha}$$

Consider $\mathcal{F}\colon \mathfrak{P}(\mathbb{N}) \to \mathfrak{P}(\mathbb{N})$ monotone. Then

$$\mu X\mathcal{F}(X) := \bigcap\{\mathfrak{X} \mid \mathcal{F}(\mathfrak{X}) \subset \mathfrak{X}\}$$

is the least fixed point of $\mathcal{F}$. In other words

$$n \in \mu X\mathcal{F}(X) \text{ iff } \forall\mathfrak{X}.\underbrace{(\forall x.x \in \mathcal{F}(\mathfrak{X}) \to x \in \mathfrak{X})}_{\mathcal{F}(\mathfrak{X})\subset\mathfrak{X}} \to n \in \mathfrak{X}$$

Using the usual correspondence between HA^2 and system F we get the type

$$\forall\alpha.(\mathcal{F}(\alpha) \to \alpha) \to \alpha$$

$$\boxed{\forall\alpha.(\mathcal{F}(\alpha) \to \alpha) \to \alpha}$$

- Note that this is a closed type.

  ...and $\mathcal{F}$ is built from inductive types,

  i.e., from *closed* types and α by $\to$

$\therefore$ quantifications do not interleave

- counting levels we get the following grammar

$$\rho_n, \sigma_n \ ::= \ \alpha_n \mid \rho_n \to \sigma_n \mid \forall\alpha_\ell.\rho_\ell \qquad\qquad (\ell < n)$$

Constructive understanding of $\forall\alpha_\ell.\rho_\ell$

$$\rho_n, \sigma_n \;::=\; \alpha_n \mid \rho_n \to \sigma_n \mid \forall\alpha_\ell.\rho_\ell \qquad (\ell < n)$$

- This allows to formalise all inductive types.

$\triangleright$ but did we get more?

$\rightsquigarrow$ Aim: understand $\forall\alpha_\ell.\rho_\ell$ as inductive type

- roughly: normal inhabitants
- So a term of type $(\forall\alpha_\ell.\rho_\ell) \to \tau$ is something that can cope with all normal terms of type $(\forall\alpha_\ell.\rho_\ell)$ to produce something of type τ.

! fits: those terms usually can also have type $\rho_\ell[\alpha{:=}\sigma]$.

. . . can *usually* also. . .

- Is $\forall\mathfrak{X}/\forall\alpha$-elimination admissible?

  . . . just replace all occurences of $\mathfrak{X}$ by $\mathcal{A}$?

$\triangleright$ But what about assumptions/free variables? A proof of $\forall\mathfrak{X}.A$ "by assumption" does *not* give any information how A was derived uniformly in $\mathfrak{X}$.

$\rightsquigarrow$ why not "canonical proofs", from *proper sets of assumptions*.

The Ω-rule

- "canonical proofs", from *proper sets of assumptions*

$\therefore$ allow $\forall\mathfrak{X}_k$ only negatively (and $\forall\mathfrak{X}_\ell$ with $\ell < k$ arbitrarily)

$\rightsquigarrow \mathrm{Neg}^2_k$

$$\frac{\Gamma \vdash_n \forall\mathfrak{X}_k A \qquad \overbrace{\Delta}^{\subset \mathrm{Neg}^2_k} \vdash_k \forall\mathfrak{X}_k A \Rightarrow \Gamma, \Delta \vdash_n C}{\Gamma \vdash_n C} \quad k < n$$

- strictly positive inductive definition
- now $\forall\mathfrak{X}$-elimination a derived rule:

  $\Delta \vdash_k \forall\mathfrak{X}_k A$ proofs are *by introduction*

Collapsing

$$\frac{\Gamma \vdash_n \forall \mathfrak{X}_k A \qquad \overbrace{\Delta}^{\subset \mathrm{Neg}_k^2} \vdash_k \forall \mathfrak{X}_k A \Downarrow \quad \Gamma, \Delta \vdash_n C}{\Gamma \vdash_n C} \qquad k < n$$

- Soundness?

  . . . or: how to get rid of this rule?

- Assume that both Γ and C are arithmetical formulae

- Then induction on the derivation shows that it can be collapsed to $\vdash_0$

  . . . which doesn't use the Ω-rule!

Truth Predicates

Connection between semi-formal proofs and $\mathrm{HA}^2_{n^*+1}$ proofs?

$\rightsquigarrow$ Define (partial) truths predicates, $\forall\mathfrak{X}$ as provability

- $f \Vdash \Gamma \sqsupset C \quad \Leftrightarrow \quad f \vdash_{n^*} \Gamma \sqsupset C$
  if C atomic or of the form $\forall\mathfrak{X}A$
- $f \Vdash \Gamma \sqsupset A \to B \Leftrightarrow \forall g \in \mathbb{N}, \Delta \subset \mathrm{Neg}^2_{n^*+1}$
  $(g \Vdash \Delta \sqsupset A \Rightarrow \{f\}\left(\left\langle g, \ulcorner \Delta \urcorner \right\rangle\right) \Vdash \Gamma, \Delta \sqsupset B)$
- $\ldots$

Goal: $\quad \mathrm{HA}^2_{n^*+1}, \overrightarrow{A} \vdash A \quad \Rightarrow \quad \ldots \Vdash \overrightarrow{A} \sqsupset A$

handling of $\forall\mathfrak{X}$

$f \Vdash \Gamma \sqsupset C \quad \Leftrightarrow \quad f \vdash_{n^*} \Gamma \sqsupset C$
if C atomic or of the form $\forall\mathfrak{X}A$

Fits with our idea.

- If $d \vdash_n \Gamma \sqsupset A$ then $\ldots \vdash_n \Gamma\Sigma \sqsupset A\Sigma$

$\rightsquigarrow$ If $d \vdash_k \Delta \sqsupset \forall\mathfrak{X}_k A(\mathfrak{X}_k)$ with $\Delta \subset \mathrm{Neg}^2_k$ then $\ldots \vdash_k \Gamma \sqsupset A(\mathcal{A})$

So the following "NbE like" connection does the job.

- $e \vdash_{n^*} \Gamma \sqsupset C$, e no introduction, $C \in \mathrm{Neg}^2_{n^*} \quad \Rightarrow \quad \mathfrak{f}^C_\Gamma(e) \Vdash \Gamma \sqsupset C$
- $f \Vdash \Gamma \sqsupset C$, $C \in \mathrm{Pos}^2_{n^*} \quad \Rightarrow \quad \mathfrak{g}^C_\Gamma(f) \vdash_{n^*} \Gamma \sqsupset C$

And we get

- $\mathrm{HA}^2_{n^*}, \overrightarrow{A} \vdash A$ and $f_i \Vdash \Gamma \sqsupset A_i \quad \Rightarrow \quad \mathfrak{f}(\ldots) \Vdash \Gamma \sqsupset A$

Overview

Via embedding/cut elimination we have

$$\begin{array}{rcl} & & \mathfrak{X}_0 \text{ only for } \mathbb{N}(t) \\ \mathrm{ID}_n & \hookrightarrow & \mathrm{HA}^2_{n+1} \\ \mathrm{ID}_n & \leftarrow & \mathrm{HA}^2_{n+1} \\ \text{strictly positive} & & \end{array}$$

Via realisability/normalisation proofs we also have

$$\begin{array}{rcl} \mathrm{HA}^2_{n+1} & \sim & \mathrm{F}_{n+1} \\ \mathrm{ID}_n & \sim & \mathcal{O}_n \end{array}$$