On Fragments of Analysis with Strengths of Finitely Iterated Inductive Definitions

Klaus Aehlig

Inductive Types and System F

- Typical data structures:
  - natural numbers, lists, trees, ...

! all *inductive* types

- uniform framework: *polymorphism* (system F)
  - $\forall\alpha.(\alpha \to \alpha) \to \alpha \to \alpha$
  - $\forall\alpha.(\alpha \times \tau) \to \alpha \to \alpha$
  - $\forall\alpha.((\tau \to \alpha) \to \alpha) \to \alpha \to \alpha, \ldots$

$\triangleright$ but system F is far stronger (full second order arithmetic)!!

$\leadsto$ analyse the fragment actually needed

$$\boxed{\mu X \mathcal{F}(X) = \forall\alpha.(\mathcal{F}(\alpha) \to \alpha) \to \alpha}$$

Consider $\mathcal{F} \colon \mathfrak{P}(\mathbb{N}) \to \mathfrak{P}(\mathbb{N})$ monotone. Then

$$\mu X \mathcal{F}(X) := \bigcap\{\mathfrak{X} \mid \mathcal{F}(\mathfrak{X}) \subset \mathfrak{X}\}$$

is the least fixed point of $\mathcal{F}$. In other words

$$n \in \mu X \mathcal{F}(X) \text{ iff } \forall\mathfrak{X}.\, \underbrace{(\forall x.x \in \mathcal{F}(\mathfrak{X}) \to x \in \mathfrak{X})}_{\mathcal{F}(\mathfrak{X}) \subset \mathfrak{X}} \to n \in \mathfrak{X}$$

Using the usual correspondence between HA^2 and system F we get the type

$$\forall\alpha.(\mathcal{F}(\alpha) \to \alpha) \to \alpha$$

$$\boxed{\forall\alpha.(\mathcal{F}(\alpha) \to \alpha) \to \alpha}$$

- Note that this is a closed type.

  ...and $\mathcal{F}$ is built from inductive types,

  i.e., from *closed* types and α by $\to$

$\therefore$ quantifications do not interleave

- counting levels we get the following grammar

$$\rho_n, \sigma_n \ ::= \ \alpha_n \mid \rho_n \to \sigma_n \mid \forall\alpha_\ell.\rho_\ell \qquad\qquad (\ell < n)$$

Constructive understanding of $\forall\alpha_\ell.\rho_\ell$

$$\rho_n, \sigma_n \ ::= \ \alpha_n \mid \rho_n \to \sigma_n \mid \forall\alpha_\ell.\rho_\ell \qquad\qquad (\ell < n)$$

- This allows to formalise all inductive types.

$\triangleright$ but did we get more?

$\rightsquigarrow$ Aim: understand $\forall\alpha_\ell.\rho_\ell$ as inductive type

- roughly: normal inhabitants
- So a term of type $(\forall\alpha_\ell.\rho_\ell) \to \tau$ is something that can cope with all normal terms of type $(\forall\alpha_\ell.\rho_\ell)$ to produce something of type τ.

! fits: those terms *usually* can also have type $\rho_\ell[\alpha{:=}\sigma]$.

The Ω-rule, yet another motivation...

- $\forall\mathfrak{X}$-elimination lacks the sub-formula property
- For $\mathbb{N}$-induction the solution was the ω-rule

  (making induction admissible)
- Is $\forall\mathfrak{X}$-elimination admissible?

  ...just replace all occurences of $\mathfrak{X}$ by $\mathcal{A}$?

$\triangleright$ But what about assumptions? A proof of $\forall\mathfrak{X}.A$ "by assumption" does *not* give any information how A was derived uniformly in $\mathfrak{X}$.

- In the ω-rule only "canonical terms" where used...

$\rightsquigarrow$ why not "canonical proofs", from *proper sets of assumptions.*

The Ω-rule

- "canonical proofs", from *proper sets of assumptions*

$\therefore$ allow $\forall\mathfrak{X}_k$ only negatively (and $\forall\mathfrak{X}_\ell$ with $\ell < k$ arbitrarily)

$\rightsquigarrow \mathrm{Neg}^2_k$

$$\dfrac{\Gamma \vdash_n \forall\mathfrak{X}_k A \qquad \begin{array}{c} \overbrace{\Delta}^{\subset \mathrm{Neg}^2_k} \vdash_k \forall\mathfrak{X}_k A \\ \Downarrow \\ \Gamma, \Delta \vdash_n C \end{array}}{\Gamma \vdash_n C} \quad k < n$$

- strictly positive inductive definition
- now $\forall\mathfrak{X}$-elimination a derived rule:

  $\Delta \vdash_k \forall\mathfrak{X}_k A$ proofs are *by introduction*

$\boxed{\forall \mathfrak{X} A(\mathfrak{X})$ **by introduction**$}$

$$
\dfrac{\begin{array}{c}\Gamma \\ \vdots \\ A(\mathfrak{X})\end{array}}{\forall \mathfrak{X} A(\mathfrak{X})}
\quad
\begin{array}{c}\cdot[\mathfrak{X}{:=}\mathcal{A}] \\ \rightsquigarrow \\ (\mathfrak{X}\not\in \mathrm{FV}(\Gamma))\end{array}
\quad
\begin{array}{c}\Gamma \\ \vdots \\ A(\mathcal{A})\end{array}
$$

Collapsing

$$\frac{\Gamma \vdash_n \forall \mathfrak{X}_k A \qquad \begin{array}{c} \overbrace{\Delta}^{\subset \mathrm{Neg}^2_k} \vdash_k \forall \mathfrak{X}_k A \\ \Downarrow \\ \Gamma, \Delta \vdash_n C \end{array}}{\Gamma \vdash_n C} \quad k < n$$

- Soundness?

  ... or: how to get rid of this rule?

- Assume that both Γ and C are arithmetical formulae

- Then induction on the derivation shows that it can be collapsed to $\vdash_0$

  ... which doesn't use the Ω-rule!

Semi-formal proofs

So we have a semi formal notion $\cdot \vdash_n \cdot \sqsupset \cdot$ of proofs (for $n < n^* + 1$), given by lots of rules.

> (Axiom), (Assumption), ($\rightarrow$-elim), ($\wedge$-elim), ($\forall x$-elim) *provided the main premise is no introduction*
>
> $\vdots$
>
> (Ω_{k+1}) if $k < n$ and $d \vdash_n \Gamma \sqsupset \forall \mathfrak{X}_k A(\mathfrak{X}_k)$ and
>
> (for all $\Delta \subset \mathrm{Neg}^2_k$ it holds that if $d' \vdash_k \Delta \sqsupset \forall \mathfrak{X}_k A(\mathfrak{X}_k)$ then $\{e\}\,(\ldots d' \ldots) \vdash_n \Delta, \Gamma \sqsupset B$)
>
> then $\langle \ldots, d, e \rangle \vdash_n \Gamma \sqsupset B$

We also have the formal notion of proofs in $\mathrm{HA}^2_{n^*+1}$. *What is the connection?*

Truth Predicates

Connection between semi-formal proofs and $\mathrm{HA}^2_{n^*+1}$ proofs?

$\rightsquigarrow$ Define (partial) truths predicates, $\forall\mathfrak{X}$ as provability

- $f \Vdash \Gamma \sqsupset C \quad \Leftrightarrow \quad f \vdash_{n^*} \Gamma \sqsupset C$
  if C atomic or of the form $\forall\mathfrak{X} A$
- $f \Vdash \Gamma \sqsupset A \to B \Leftrightarrow \forall g \in \mathbb{N}, \Delta \subset \mathrm{Neg}^2_{n^*+1}$
  $(g \Vdash \Delta \sqsupset A \Rightarrow \{f\}\left(\left\langle g, \ulcorner\Delta\urcorner\right\rangle\right) \Vdash \Gamma, \Delta \sqsupset B)$
- $\ldots$

Goal: $\quad \mathrm{HA}^2_{n^*+1}, \overrightarrow{A} \vdash A \quad \Rightarrow \quad \ldots \Vdash \overrightarrow{A} \sqsupset A$

$\boxed{\textbf{handling of } \forall\mathfrak{X}}$

$$f \Vdash \Gamma \sqsupset C \quad \Leftrightarrow \quad f \vdash_{n^*} \Gamma \sqsupset C$$
if C atomic or of the form $\forall\mathfrak{X} A$

Fits with our idea.

- If $d \vdash_n \Gamma \sqsupset A$ then $\ldots \vdash_n \Gamma\Sigma \sqsupset A\Sigma$

$\rightsquigarrow$ If $d \vdash_k \Delta \sqsupset \forall\mathfrak{X}_k A(\mathfrak{X}_k)$ with $\Delta \subset \mathrm{Neg}^2_k$ then $\ldots \vdash_k \Gamma \sqsupset A(\mathcal{A})$

So the following "NbE like" connection does the job.

- $e \vdash_{n^*} \Gamma \sqsupset C$, e no introduction, $C \in \mathrm{Neg}^2_{n^*} \quad \Rightarrow \quad \mathfrak{f}^C_\Gamma(e) \Vdash \Gamma \sqsupset C$
- $f \Vdash \Gamma \sqsupset C$, $C \in \mathrm{Pos}^2_{n^*} \quad \Rightarrow \quad \mathfrak{g}^C_\Gamma(f) \vdash_{n^*} \Gamma \sqsupset C$

And we get

- $\mathrm{HA}^2_{n^*}, \overrightarrow{A} \vdash A$ and $f_i \Vdash \Gamma \sqsupset A_i \quad \Rightarrow \quad \mathfrak{f}(\ldots) \Vdash \Gamma \sqsupset A$

The consistency proof, Overview

- $\mathrm{HA}^2_{n^*}, \overrightarrow{A} \vdash A$ and $f_i \Vdash \Gamma \sqsupset A_i \quad \Rightarrow \quad \mathfrak{f}(\dots) \Vdash \Gamma \sqsupset A$
- In particular: $\mathrm{HA}^2_{n^*} \vdash 1 = 0 \quad \Rightarrow \quad \dots \Vdash \emptyset \sqsupset 1 = 0$
- Given that $1 = 0$ is atomic, we obtain $\dots \vdash_{n^*} \emptyset \sqsupset 1 = 0$.
- Collapsing yields $\dots \vdash_0 \emptyset \sqsupset 1 = 0$.
- But this is a normal proof, first order arithmetic, without induction!
- Hence $1 = 0$ is true — absurd!

Leveling

- Let $\mathrm{HA}^2_{\widehat{\mathfrak{X}}}$ be HA^2, restricted to $\widehat{\mathfrak{X}}$ as the *only* set variable
- Looking at the binding position we get

$$\mathrm{HA}^2_{\widehat{\mathfrak{X}}} \vdash A \quad \Rightarrow \quad \mathrm{HA}^2_n \vdash A \text{ for some } n$$

$\leadsto$ simple characterisation of $\mathrm{ID}_{<\omega}$.

- However, doesn't extend, as...

$$\begin{aligned}
\Phi^{\alpha}_{\widehat{\mathfrak{X}}}(x) &= \exists y. y \sim x \wedge \widehat{\mathfrak{X}}\left\langle \ulcorner \alpha \urcorner, y \right\rangle \\
\Phi^{\rho\to\sigma}_{\widehat{\mathfrak{X}}}(x) &= \forall y. \Phi^{\rho}_{\widehat{\mathfrak{X}}}(y) \to \Phi^{\sigma}_{\widehat{\mathfrak{X}}}(\ulcorner \dot{x}\dot{y} \urcorner) \\
\Phi^{\forall\alpha\rho}_{\widehat{\mathfrak{X}}}(x) &= \forall \widehat{\mathfrak{Y}}. (\forall z. (z)_0 \neq \ulcorner \alpha \urcorner \to (\widehat{\mathfrak{X}} z \leftrightarrow \widehat{\mathfrak{Y}} z)) \to \Phi^{\rho}_{\widehat{\mathfrak{Y}}}(x)
\end{aligned}$$

...are the truth predicates for *full* system F.

Summary

Via embedding/cut elimination we have

$$
\begin{array}{rcl}
 & & \mathfrak{X}_0 \text{ only for } \mathbb{N}(t) \\
\mathrm{ID}_n & \hookrightarrow & \mathrm{HA}^2_{n+1} \\
\mathrm{ID}_n & \leftarrow & \mathrm{HA}^2_{n+1} \\
\text{strictly positive} & &
\end{array}
$$

Via realisability/normalisation proofs we also have

$$
\begin{array}{lcl}
\mathrm{HA}^2_{n+1} & \sim & \mathrm{F}_{n+1} \\
\mathrm{ID}_n & \sim & \mathcal{O}_n
\end{array}
$$