A syntactical analysis of non-size-increasing polynomial time computation

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Example: $\texttt{insertion sort}$ & $x \mapsto 2^x$

$$
\begin{aligned}
\texttt{sort}([]) &= [] \\
\texttt{sort}([a \mid l]) &= \texttt{insert}(a, \texttt{sort}(l))
\end{aligned}
\qquad
\begin{aligned}
\texttt{exp}(0) &= S0 \\
\texttt{exp}(Sx) &= \texttt{double}(\texttt{exp}(x))
\end{aligned}
$$

$$
\begin{aligned}
\texttt{insert}(a, []) &= [a] \\
\texttt{insert}(a, [b \mid \ell]) &= [b' \mid \texttt{insert}(a', \ell)]
\end{aligned}
\qquad
\begin{aligned}
\texttt{double}(0) &= 0 \\
\texttt{double}(Sx) &= S(S(\texttt{double}(x)))
\end{aligned}
$$

a', b' a permutation of a, b with $b' \leq a'$

Controlling the size of the recursive values?

- "They're small because everything is small!"
- Therefore only non-size-increasing functions are considered.
- We need a *syntactical* bound on the "size" of every term of an inductive type (e.g. numbers, lists, ...).
- Non-growing syntactical measure?!

$$t \to t' \Longrightarrow \mathrm{FV}\,(t) \supset \mathrm{FV}\,(t')$$

$\boxed{\textbf{Defining a term system out of this idea?}}$

- **first step:** Enforce free variables
  - define a new type $\diamond$ with

$$t^{\diamond} \Longrightarrow \mathrm{FV}\,(t) \neq \emptyset$$

  - successor of Type

$$\diamond \multimap \mathbf{N} \multimap \mathbf{N}$$

- **problem:**

$$\mathbf{S}_1 d^\diamond (\mathbf{S}_1 d^\diamond (\ldots (\mathbf{S}_1 d^\diamond 0)))$$

- **second step:** *linear* typing, i.e. only form (st) if

$$\mathrm{FV}\,(s) \cap \mathrm{FV}\,(t) = \emptyset$$

Overview of the term system

- **Abstractions** (Introduced by special syntax)
  - λ-abstraction: $\lambda x\, t$
  - Pair: $\langle s, t \rangle$
- **Data types:** (Introduced by constants)
  - Binary numerals: 0, $\mathbf{S}_0$, $\mathbf{S}_1$
  - Lists: $\mathbf{nil}_\tau$, $\mathbf{cons}_\tau$
  - Tensor: $\otimes_{\tau,\rho}$
  - Booleans: $\mathbf{t}$, $\mathbf{f}$

Overview of the term system (cont'd)

- **Eliminations** written as application ("Vector Notation")
  - Application: $t^{\tau \multimap \rho} s$
  - Iteration: $t^{\mathbf{L}(\tau)} \{s\}\, r$

    $t^{\mathbf{N}} \{s_0, s_1\}\, r$
  - Projections: $t^{\tau \times \rho}\mathbf{t}$ and $t^{\tau \times \rho}\mathbf{f}$
  - Tensor-Elimination: $t^{\tau \otimes \rho}(\lambda x^{\tau}, y^{\rho}\, s)$
  - Cases: $t^{\mathrm{bool}} \langle s_0, s_1 \rangle$
- Type $\diamond$: $x^{\diamond}$

Terms and Types

- $\tau, \rho ::= \ \diamond \mid \mathbf{N} \mid \mathrm{bool} \mid \tau \multimap \rho \mid \tau \otimes \rho \mid \tau \times \rho \mid \mathbf{L}(\tau)$
- $t, s ::= \ x \mid c \mid \lambda x^{\tau} t \mid \langle t, s \rangle \mid ts \mid \{t\} \mid \{t, s\}$
- Constants:

$\mathbf{t}$	bool
$\mathbf{f}$	bool
$\mathbf{1}$	$\mathbf{N}$
$\mathbf{S}_0$	$\diamond \multimap \mathbf{N} \multimap \mathbf{N}$
$\mathbf{S}_1$	$\diamond \multimap \mathbf{N} \multimap \mathbf{N}$
$\mathbf{nil}_\tau$	$\mathbf{L}(\tau)$
$\mathbf{cons}_\tau$	$\diamond \multimap \tau \multimap \mathbf{L}(\tau) \multimap \mathbf{L}(\tau)$
$\otimes_{\tau,\rho}$	$\tau \multimap \rho \multimap \tau \otimes \rho$

Linear Typing (λ-Terms)

$$\frac{\Gamma \cup \{x^\tau\} \vdash t^\rho}{\Gamma \vdash (\lambda x^\tau\, t)^{\tau \multimap \rho}} \qquad (\multimap^+)$$

$$\frac{\Gamma_1 \vdash t^{\tau \multimap \rho} \qquad \Gamma_2 \vdash s^\tau}{\Gamma_1, \Gamma_2 \vdash (ts)^\rho} \qquad (\multimap^-)$$

$$(\lambda x^\tau\, t)s \mapsto t[s/x]$$

Linear Typing (The pair)

$$\frac{\Gamma \vdash t^{\tau} \qquad \Gamma \vdash s^{\rho}}{\Gamma \vdash \langle t, s \rangle^{\tau \times \rho}} \qquad (\times^{+})$$

$$\frac{\Gamma \vdash t^{\tau \times \rho}}{\Gamma \vdash (t\mathbf{t})^{\tau}} \qquad (\times_1^{-})$$

$$\frac{\Gamma \vdash t^{\tau \times \rho}}{\Gamma \vdash (t\mathbf{f})^{\rho}} \qquad (\times_0^{-})$$

$$\langle t, s \rangle\, \mathbf{t} \mapsto t$$

$$\langle t, s \rangle\, \mathbf{f} \mapsto s$$

Linear Typing (Iteration)

$$\frac{\Gamma \vdash t^{\mathbf{L}(\tau)} \qquad \emptyset \vdash s^{\diamond \multimap \tau \multimap \rho \multimap \rho}}{\Gamma \vdash (t\,\{s\})^{\rho \multimap \rho}} \qquad (\mathbf{L}(\tau)^{-})$$

$$\mathbf{nil}_{\tau}\,\{s\}\,r \mapsto r$$

$$\mathbf{cons}_{\tau} d^{\diamond} a \ell\,\{s\}\,t \mapsto t d^{\diamond} a (\ell\,\{s\}\,t) \quad \ell \text{ a list}$$

Reductions

$$\frac{t \mapsto t'}{t \to t'} \qquad \frac{t \to t'}{ts \to t's} \qquad \frac{s \to s'}{ts \to ts'}$$

- $\Gamma_1 \cup \{x^\rho\} \vdash t^\tau$; $\Gamma_2 \vdash s^\rho$; Γ_1, Γ_2 disjoint $\Longrightarrow \Gamma_1, \Gamma_2 \vdash t[s/x^\rho]^\tau$
- $\Gamma \vdash t^\tau$; $t \to t' \Longrightarrow \Gamma \vdash t'^\tau$
- Every normal, almost closed term of type $\mathbf{N}$ is a numeral. Every normal, almost closed term of type $\mathbf{L}(\tau)$ is a list. Every normal, almost closed term of type $\diamond$ is a variable.

Length

- Inductive Definition of $|t|$:

$$|c| = |x| = 1$$

$$|ts| = |t| + |s|$$

$$|\lambda x^\tau\, s| = |s| + 1$$

$$|\langle t, s\rangle| = \max\{|t|\,,|s|\} + 1$$

$$|\{t\}| = |\{t_1, t_2\}| = 0$$

- $\Gamma_1 \vdash t^\tau$, $\Gamma_2 \vdash s^\rho$; Γ_1, Γ_2 disjoint $\Longrightarrow |t[s/x]| \leq |t| + |s|$

  In particular, $|(\lambda x^\tau\, t)s| > |t[s/x]|$.

Canonical Reducts

$$(\lambda x\, t)s \quad \to^1 \quad t[s/x]$$

$$\mathbf{cons}_\tau d_1^\diamond r_1^\tau (\ldots(\mathbf{cons}_\tau d_n^\diamond r_n^\tau \mathbf{nil}_\tau))\,\{h\}\,t \quad \to^{n+1} \quad h d_1^\diamond r_1^\tau (\ldots(h d_n^\diamond r_n^\tau t))$$

$$\mathbf{S}_{i_1} r_1^\diamond (\ldots(\mathbf{S}_{i_n} r_n^\diamond 1))\,\{h_0, h_1\}\,t \quad \to^{n+1} \quad h_{i_1} r_1^\diamond (\ldots(h_{i_n} r_n^\diamond t))$$

$$\langle s, t\rangle\,\mathbf{t} \quad \to^1 \quad s$$

Note that n is bound by the number of free variables!

The Polynomials

- Let $\mathbb{N}[X]$ of polynomials with natural number as coefficients be equipped with the partial ordering $\preccurlyeq$ defined by $\sum a_i X^i \preccurlyeq \sum b_i X^i \iff \forall i. a_i \leq b_i$.
- For each (not necessarily typed) term t we define a polynomial $\vartheta(t) \in \mathbb{N}[X]$ by induction on t.

$$\vartheta(x) = \vartheta(c) = 0$$

$$\vartheta(ts) = \vartheta(t) + \vartheta(s)$$

$$\vartheta(\lambda x^{\tau}\, t) = \vartheta(t)$$

$$\vartheta(\langle t, s \rangle) = \sup_{\preccurlyeq} \{t, s\}$$

$$\vartheta(\{t\}) = X \cdot \vartheta(t) + X \cdot |t|$$

$$\vartheta(\{t, s\}) = X \cdot \sup_{\preccurlyeq} \{\vartheta(t), \vartheta(s)\} + X \cdot \max\{|t|, |s|\}$$

The Theorem

Assume $\Gamma \vdash t^\tau$ and $N \geq |\mathrm{FV}(t)|$ for some $N \in \mathbb{N}$. Then there are $t = t_0 \to t_1 \to \ldots \to t_k$ with t_k normal and

$$\vartheta(t_k)(N) + |t_k| + k \leq \vartheta(t)(N) + |t|$$

Proofed by induction on $\vartheta(t)(N) + |t|$.

insertion sort (again)

Let τ be a type equipped with a linear ordering. Assume that this ordering is represented by the term $\lhd_\tau$ of type $\tau \multimap \tau \multimap \text{bool} \otimes (\tau \otimes \tau)$, i.e.
$\lhd_\tau ts \rightarrow^* \otimes_{\text{bool},\tau\otimes\tau}\mathbf{t}(\otimes_{\tau,\tau}ts)$, if s is "smaller" than t, and
$\lhd_\tau ts \rightarrow^* \otimes_{\text{bool},\tau\otimes\tau}\mathbf{f}(\otimes_{\tau,\tau}ts)$ otherwise.

Using this function, we can define a sort function of type $\tau \multimap \tau \multimap \tau \otimes \tau$ for two elements, i.e. a function taking two arguments and returning them in the correct order:

$$\lhd'_\tau = \lambda p_1^\tau, p_2^\tau \ \lhd_\tau p_1 p_2 (\lambda y^{\text{bool}}, p^{\tau\otimes\tau}\ p(\lambda p_1^\tau, p_2^\tau \\ y \left\langle \otimes_{\tau,\tau} p_1 p_2, \otimes_{\tau,\tau} p_2 p_1 \right\rangle))$$

With this function we define

$$\begin{aligned}\texttt{insert} = \lambda l \quad & l\{\lambda x_1^{\diamond}, y_1^{\tau}, p^{\diamond \multimap \tau \multimap \mathbf{L}(\tau)}, x_2^{\diamond}, y_2^{\tau} \\ & \lhd'_{\tau} y_1 y_2 (\lambda z_1^{\tau}, z_2^{\tau}\ \mathbf{cons}_{\tau} x_1 z_1 (p x_2 z_2))\} \\ & (\lambda x^{\diamond}, y^{\tau}\ \mathbf{cons}_{\tau} x y \mathbf{nil}_{\tau})\end{aligned}$$

And finally:

$$\begin{aligned}\texttt{sort} = \lambda l^{\tau}\ l & \left\{\lambda x^{\diamond}, y^{\tau}, l^{\mathbf{L}(\tau)}\ \texttt{insert} l x y\right\} \\ & \mathbf{nil}_{\tau}\end{aligned}$$

Counting braces, we get $\vartheta\,(\texttt{insertionsort}) = X \cdot \vartheta\,(\texttt{insert}) + O(X) = X^2 \cdot \vartheta\,(\lhd'_{\tau}) + O(X^2) = X^2 \cdot \vartheta\,(\lhd_{\tau}) + O(X^2)$, as $\vartheta\,(\lhd'_{\tau}) = \vartheta\,(\lhd_{\tau})$.