Semantics of First-Order Logic

Let $\mathcal{A} = (A, R_1^{\mathcal{A}}, \ldots)$ and $\eta\colon V \to A$.
Define $\mathcal{A}, \eta \models \varphi$ for $\varphi \in \mathsf{FOL}[R_1, \ldots]$ inductively.

- $\mathcal{A}, \eta \models x = y$ iff $\eta(x) = \eta(y)$
- $\mathcal{A}, \eta \models R_1 y_1 \ldots y_n$ iff $(\eta(y_1), \ldots, \eta(y_n)) \in R_i^{\mathcal{A}}$
- $\mathcal{A}, \eta \models \varphi \wedge \psi$ iff $\ldots$
- $\mathcal{A}, \eta \models \forall x \varphi$ iff for all $a \in A$ we have $\mathcal{A}, \eta_x^a \models \varphi$
- $\mathcal{A}, \eta \models \exists x \varphi$ iff for some $a \in A$ we have $\mathcal{A}, \eta_x^a \models \varphi$

Negation Normal Form

Lemma.

$\mathcal{A}, \eta \models \neg\forall x\varphi$ iff $\mathcal{A}, \eta \models \exists x\neg\varphi$

$\mathcal{A}, \eta \models \neg\exists x\varphi$ iff $\mathcal{A}, \eta \models \forall x\neg\varphi$

Recall from propositional logic.

$\mathcal{A}, \eta \models \neg(\varphi \wedge \psi)$ iff $\mathcal{A}, \eta \models (\neg\varphi) \vee (\neg\psi)$

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$\mathcal{A}, \eta \models \neg\neg\varphi$ iff $\mathcal{A}, \eta \models \varphi$

Prenex Normal Form

Lemma. Assume $x \notin \mathsf{fv}(\psi)$.

$\mathcal{A}, \eta \models (\forall x \varphi) \wedge \psi$ iff $\mathcal{A}, \eta \models \forall x (\varphi \wedge \psi)$

$\mathcal{A}, \eta \models (\forall x \varphi) \vee \psi$ iff $\mathcal{A}, \eta \models \forall x (\varphi \vee \psi)$

$\mathcal{A}, \eta \models (\exists x \varphi) \wedge \psi$ iff $\mathcal{A}, \eta \models \exists x (\varphi \wedge \psi)$

$\mathcal{A}, \eta \models (\exists x \varphi) \vee \psi$ iff $\mathcal{A}, \eta \models \exists x (\varphi \vee \psi)$

Words—Spot the difference!

a a b a a *vs* a a a a a

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a a b a a *vs* a a a a b

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a b a c a *vs* a a b c a

Ehrenfeucht-Fraïssé Games

The game is played on configurations

$$\underbrace{(A, R_1^{\mathcal{A}}, R_2^{\mathcal{A}}, \ldots)}_{\mathcal{A}}, a_1, \ldots, a_k \;\mid\; \underbrace{(B, R_1^{\mathcal{B}}, R_2^{\mathcal{B}}, \ldots)}_{\mathcal{B}}, b_1, \ldots, b_k$$

where $a_1, \ldots, a_k \in A$ and $b_1, \ldots, b_k \in B$.

In each round, Spoiler picks either $a_{k+1} \in A$ or $b_{k+1} \in B$. Then Duplicator picks the other.

Duplicator needs to keep the invariants

- $a_i = a_j$ iff $b_i = b_j$
- $R_i^{\mathcal{A}}(a_{i_1}, \ldots, a_{i_\ell})$ iff $R_i^{\mathcal{B}}(b_{i_1}, \ldots, b_{i_\ell})$