First-Order Structures

Definition. A first-order structure $\mathcal{A} = (A, R_1, R_2, \ldots)$ is given by

- A set A, called the "universe" of the structure
- Relations $R_1, R_2, \ldots$ on A.
  I.e., each $R_i \subseteq A^{n_i}$ for some n_i, called the arity.

Syntax of First-Order Logic

The set $\mathsf{FOL}[R_1, \ldots, R_n]$ of first-order formulae over the relation symbols $R_1, \ldots, R_n$ is freely generated as follows.

- $(x = y)$ for variables x, y
- $R_j x_{i_1} \ldots x_{i_{n_j}}$ if R_j is of arity n_j
- $\top$, $\bot$, $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$ for formulae ϕ, ψ
- $\forall x \varphi$ and $\exists x \varphi$ for ϕ a formula and x a variable

Semantics of First-Order Logic

Let $\mathcal{A} = (A, R_1^{\mathcal{A}}, \ldots)$ and $\eta\colon V \to A$.
Define $\mathcal{A}, \eta \models \varphi$ for $\varphi \in \mathsf{FOL}[R_1, \ldots]$ inductively.

- $\mathcal{A}, \eta \models x = y$ iff $\eta(x) = \eta(y)$
- $\mathcal{A}, \eta \models R_1 y_1 \ldots y_n$ iff $(\eta(y_1), \ldots, \eta(y_n)) \in R_i^{\mathcal{A}}$
- $\mathcal{A}, \eta \models \varphi \wedge \psi$ iff $\ldots$
- $\mathcal{A}, \eta \models \forall x \varphi$ iff for all $a \in A$ we have $\mathcal{A}, \eta_x^a \models \varphi$
- $\mathcal{A}, \eta \models \exists x \varphi$ iff for some $a \in A$ we have $\mathcal{A}, \eta_x^a \models \varphi$