2.3 First-Order Logic

As mentioned, the structures we have been looking at are first-order structures. The language usually used to describe them is first-order logic. We'll see that first-order logic can, without a significant overhead, express everything we could express in linear temporal logic. However, the complexity of model checking and the satisfiability problem will turn out to be much harder for first-order logic.

Recall the general setting of the structures we're talking about.

First-Order Structures

Definition. A first-order structure $\mathcal{A} = (A, R_1, R_2, \ldots)$ is given by

- A set A, called the "universe" of the structure
- Relations $R_1, R_2, \ldots$ on A.
  I.e., each $R_i \subseteq A^{n_i}$ for some n_i, called the arity.

So far, the elements of the universe were always handled implicitly. So, we didn't have to have names for them. Now we assume some fixed infinite set

$$V = \{x_1, x_2, \ldots\}$$

of first order variables. Our notion of (first-order) formula will implicitly refer to this set.

We will use $x, y, z, \ldots$ to denote variables.

Syntax of First-Order Logic

The set $\mathsf{FOL}[R_1, \ldots, R_n]$ of first-order formulae over the relation symbols $R_1, \ldots, R_n$ is freely generated as follows.

- $(x = y)$ for variables x, y
- $R_j x_{i_1} \ldots x_{i_{n_j}}$ if R_j is of arity n_j
- $\top$, $\bot$, $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$ for formulae ϕ, ψ
- $\forall x\varphi$ and $\exists x\varphi$ for ϕ a formula and x a variable

Intuitively, we consider the variable x to be bound in $\forall x\varphi$. This intuition is reflected in our definition of the set of "free variables" of a formula.

$$\begin{array}{lcl}
\mathsf{fv}(x = y) & = & \{x, y\} \\
\mathsf{fv}(Ry_1 \ldots y_k) & = & \{y_1, \ldots, y_k\} \\
\mathsf{fv}(\top) = \mathsf{fv}(\bot) & = & \emptyset \\
\mathsf{fv}(\neg\varphi) & = & \mathsf{fv}(\varphi) \\
\mathsf{fv}((\varphi \wedge \psi)) = \mathsf{fv}((\varphi \vee \psi)) & = & \mathsf{fv}(\varphi) \cup \mathsf{fv}(\psi) \\
\mathsf{fv}(\forall x\varphi) = \mathsf{fv}(\exists x\varphi) & = & \mathsf{fv}(\varphi) \setminus \{x\}
\end{array}$$

To assign a meaning to first-order formulae, we do not only have to know, what the relation symbols stand for, but we also need to know, how the variables are to be interpreted—just as we had to know the meaning of "now" in LTL.

To do so, we use a mapping η from the variables to the universe. We also need the notion of a mapping modified at one particular place

> For $\eta\colon V \to A$, $x \in V$, $a \in A$ we set
>
> $$\eta_x^a(y) = \begin{cases} \eta(y) & x \not\equiv y \\ a & x \equiv y \end{cases}$$

Semantics of First-Order Logic

Let $\mathcal{A} = (A, R_1^{\mathcal{A}}, \ldots)$ and $\eta\colon V \to A$.
Define $\mathcal{A}, \eta \models \varphi$ for $\varphi \in \mathsf{FOL}[R_1, \ldots]$ inductively.

- $\mathcal{A}, \eta \models x = y$ iff $\eta(x) = \eta(y)$
- $\mathcal{A}, \eta \models R_1 y_1 \ldots y_n$ iff $(\eta(y_1), \ldots, \eta(y_n)) \in R_i^{\mathcal{A}}$
- $\mathcal{A}, \eta \models \varphi \wedge \psi$ iff ...
- $\mathcal{A}, \eta \models \forall x \varphi$ iff for all $a \in A$ we have $\mathcal{A}, \eta_x^a \models \varphi$
- $\mathcal{A}, \eta \models \exists x \varphi$ iff for some $a \in A$ we have $\mathcal{A}, \eta_x^a \models \varphi$

Again, we have a coincidence Lemma.

Lemma.
η, η' such that $\eta(x) = \eta'(x)$ for all $x \in \mathsf{fv}(\varphi)$.
Then $\mathcal{A}, \eta \models \varphi$ iff $\mathcal{A}, \eta' \models \varphi$.

Proof. Induction on φ.

...

Most cases are straight forward. The only non-trivial case is that of quantifiers.

...

Case $\varphi \equiv \forall x \psi$. Then $\mathsf{fv}(\psi) \subseteq \mathsf{fv}(\forall x \psi) \cup \{x\}$. Moreover, for all $a \in A$ and $y \in \mathsf{fv}(\forall x \psi) \cup \{x\}$ we have $\eta^a_x(y) = {\eta'}^a_x(y)$.
Hence, by IH,
$\mathcal{A}, \eta \models \forall x \psi$ iff
for all $a \in A$, $\mathcal{A} \eta^a_x \models \psi$ iff
for all $a \in A$, $\mathcal{A} {\eta'}^a_x \models \psi$ iff
$\mathcal{A}, \eta' \models \forall x \psi$

...

The case of $\exists x \phi$ is handled similarly.
The Coincidence Lemma justifies the following

Notation. If $\mathsf{fv}(\varphi) = \{x_1, \ldots, x_n\}$ with $x_1, \ldots, x_n$ understood, we write

$$\mathcal{A}, a_1, \ldots, a_n \models \varphi$$

if for some (and hence any) η with $\eta(x_i) = a_i$ for all i,

$$\mathcal{A}, \eta \models \varphi$$

In particular, we just write

$$\mathcal{A} \models \varphi$$

if $\mathsf{fv}(\varphi) = \emptyset$.

As usual, we use some abbreviations. Especially when one relation symbol is $<$ and is thought of as some linear order, quite a few abbreviations are commonly in use.

$<$ is usually written infix, i.e. $x < y$ denotes $< xy$.

Moreover,

$$\begin{array}{lcl} x \leq y & \equiv & (x < y \vee x = y) \\ x > y & \equiv & y < x \\ x \geq y & \equiv & y \leq x \\ \ldots & & \end{array}$$

Over discrete linear orders, we can do even more. Remember that LTL's `X`-modality required us to define "the next point in time". This concept can be described as "in the future, but between now and that point there is nothing else".

$$(\{1, \ldots, t\}, <), t, t' \models (x < x' \wedge \neg \exists y (x < y \wedge y < x'))$$

$$\text{iff}$$

$$t' = t + 1$$

This observation motivates us to abbreviate the above formula by $\chi_{\text{next}}(x, y)$.