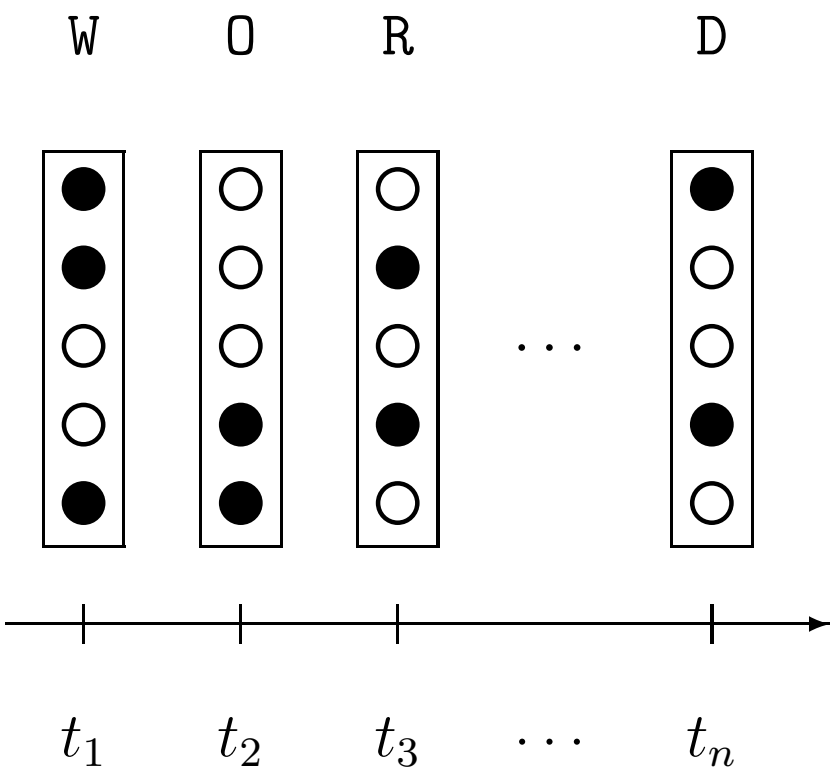
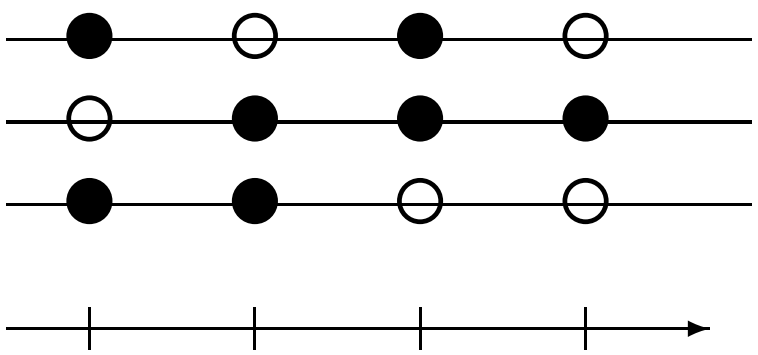
First-Order Logic



Finite Trace Structures



$R = \{1, 3\}$

$Q = \{2, 3, 4\}$

$P = \{1, 2\}$

1 2 3 4

Universe $\{1, 2, 3, 4\}$

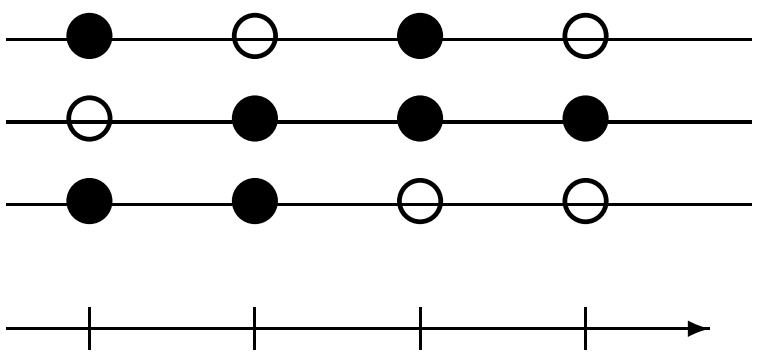
$<= \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$

First-Order Structures

Definition. A first-order structure $\mathcal{A} = (A, R_1, R_2, \ldots)$ is given by

- A set A, called the "universe" of the structure
- Finitely many relations $R_1, R_2, \ldots$ on A.
  I.e., each $R_i \subseteq A^{n_i}$ for some n_i, called the arity.

Finite Trace Structures



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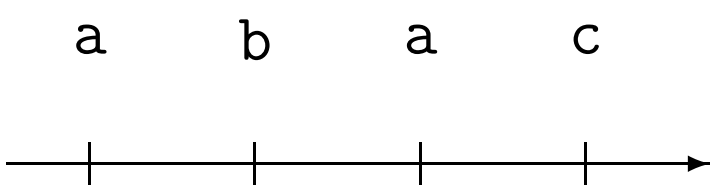
Word Structures

$P_{\mathtt{a}} = \{1, 3\}$, $P_{\mathtt{b}} = \{2\}$, $P_{\mathtt{c}} = \{4\}$



Universe $\{1, 2, 3, 4\}$

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Trace Structures and Word Structures

The universe is an initial segment of natural numbers, i.e., $A = \{1, 2, \ldots, \ell\}$ and $<$ the usual order on them.

- finite trace structures are given by arbitrary unary relations P_1, P_2, ...
- word structures are given by unary relations P_a for $a \in \Sigma$ *partitioning* the univserse

Syntax of Linear Temporal Logic

The set $\mathsf{LTL}[p_1, \ldots, p_n]$ of *LTL-formulae* is freely generated

- $\top$, $\bot$, and all $p_i \in \{p_1, \ldots, p_n\}$
- If $\varphi, \psi \in \mathsf{LTL}[p_1, \ldots, p_n]$, then also $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$.
- . . . and also
  - $\mathtt{X}\varphi$ "next"
  - $\mathtt{F}\varphi$ "finally"
  - $\mathtt{G}\varphi$ "globally"
  - $(\varphi \mathtt{U} \psi)$ "until"

Model Relation for Propositional Logic

Writing $\underline{a} \models \varphi$ for $\varphi[\underline{a}] = 1$ we obtain the following.

- $\underline{a} \models \top$ always holds and $\underline{a} \models \bot$ never holds
- $\underline{a} \models \neg\varphi$ holds iff $\underline{a} \models \varphi$ does *not* hold
- $\underline{a} \models \varphi \wedge \psi$ holds if $\underline{a} \models \varphi$ *and* $\underline{a} \models \psi$ both hold.
  $\underline{a} \models \varphi \vee \psi$ holds if $\underline{a} \models \varphi$ holds *or* $\underline{a} \models \psi$ holds.

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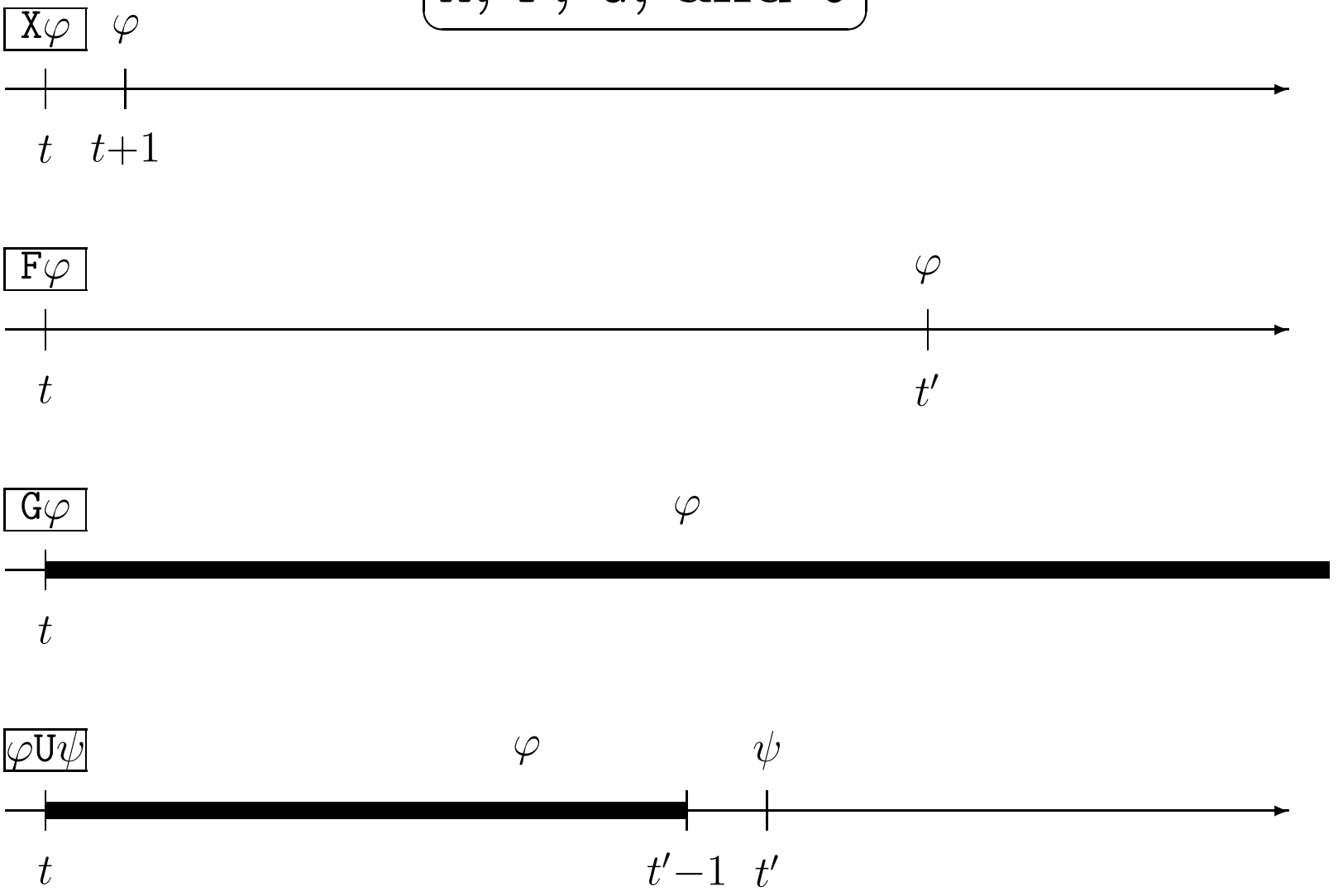
X, F, G, and U



Semantics of LTL

Let $\mathcal{A} = (\{1, \ldots, \ell\}, <, P_1, P_2, \ldots, P_n)$ and $t \in \{1, \ldots, \ell\}$.
Define $\mathcal{A}, t \models \varphi$ for $\varphi \in \mathsf{LTL}[p_1, \ldots, p_2]$ inductively.

- $\mathcal{A}, t \models \top$; $\mathcal{A}, t \not\models \bot$; $\mathcal{A}, t \models p_i$ iff $t \in P_i$
- $\mathcal{A}, t \models \varphi \wedge \psi$ iff ...
- $\mathcal{A}, t \models \mathtt{X}\varphi$ iff $t < \ell$ and $\mathcal{A}, t{+}1 \models \varphi$
- $\mathcal{A}, t \models \mathtt{F}\varphi$ iff $\mathcal{A}, t' \models \varphi$ for some $t' \geq t$
- $\mathcal{A}, t \models \mathtt{G}\varphi$ iff $\mathcal{A}, t' \models \varphi$ for all $t' \geq t$
- $\mathcal{A}, t \models \varphi \mathtt{U} \psi$ iff there is some $t' \geq t$ s.t.
  $\mathcal{A}, t' \models \psi$ and $\mathcal{A}, t'' \models \varphi$ for all $t \leq t'' < t'$

Example

For each of the following formulae, decide whether they hold at the first letter of the given words!

- a and $\mathtt{X}a$

  `baab` `abc` `aaa` `a`

- $\mathtt{F}a$

  `bbbbbba` `ba` `a`

- $\mathtt{G}a$

  `baaaaaaaaaaaaaa` `aaa` `a` `aaaaaab`

- $a\mathtt{U}b$

  `aaaaab` `abc` `bcaaab` `b`

Example

Consider the language with the predicates r_1, r_2, g_1, g_2 with the interpretation that r_1 and r_2 express that process 1 and 2, respectively, are requesting access to a shared resource, and g_1 and g_2 express that access to the shared resource is granted for process 1 and 2.

Formalise the following statements.

- "No two requests are granted at the same time."
- "Every request will eventually be granted."
- "Every request by process 1 will be granted in the next round."