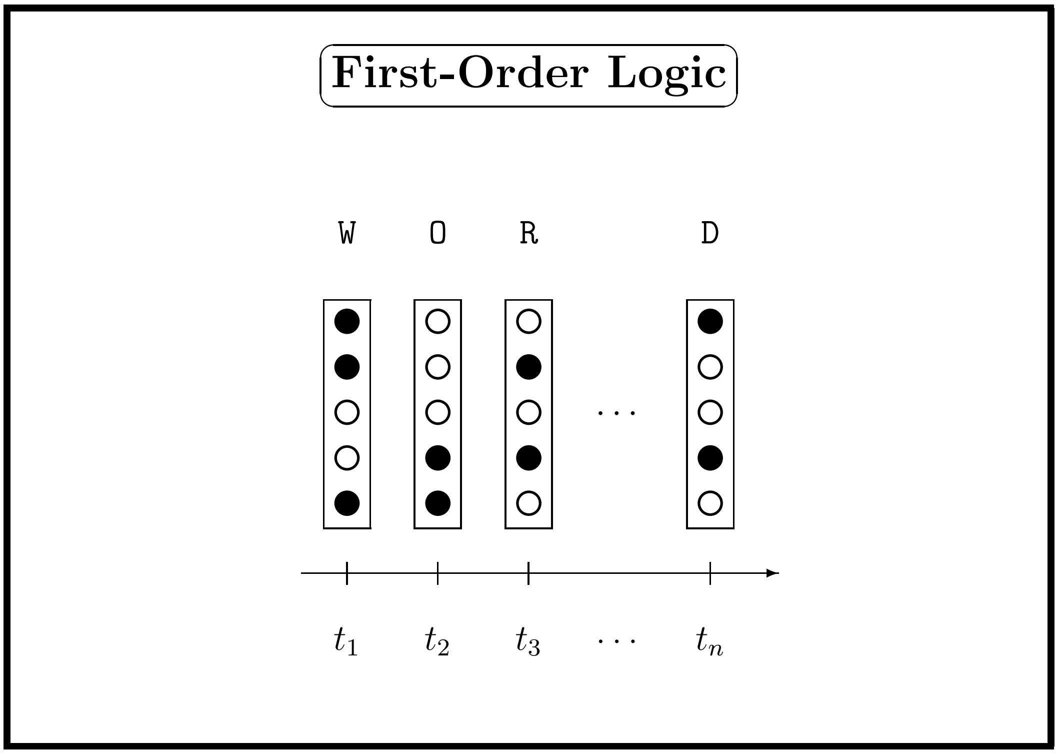
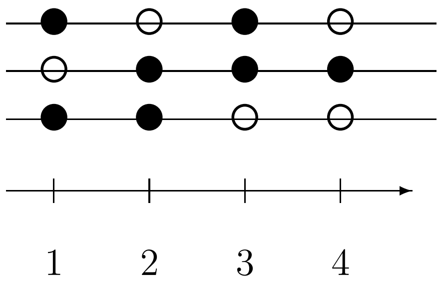
# 2 First-Order and Second-Order Logics of Strings




## 2.1 Trace Structures and Word Structures

So far, we considered propositional logic as the logic of individual bits. This was a completely static view. Now imagine a scenario where this changes over time.

## Finite Trace Structures




$R = \{1, 3\}$

$Q = \{2, 3, 4\}$

$P = \{1, 2\}$

Universe $\{1, 2, 3, 4\}$

$$< \; = \; \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$$

So, relations immediately come into play, once we consider a time component. The structures usually associated with such a setting are first-order structures.

## First-Order Structures

**Definition.** A first-order structure $\mathcal{A} = (A, R_1, R_2, \ldots)$ is given by

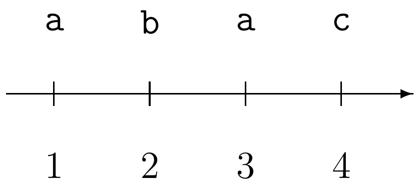
- A set $A$, called the "universe" of the structure
- Finitely many relations $R_1, R_2, \ldots$ on $A$.
  I.e., each $R_i \subseteq A^{n_i}$ for some $n_i$, called the arity.

In this course, we will mainly be looking at finite structures, that is, structures where the universe is a finite set. Nevertheless, infinite first-order structures play an important role in mathematics; in fact, most parts of logic are mainly concerned with infinite structures.

For the moment, we will not use that framework of first-order logic in its full generality. Instead, for the time being, we restrict ourself to finite trace and words.

## Word Structures

$P_{\mathtt{a}} = \{1,3\}$, $P_{\mathtt{b}} = \{2\}$, $P_{\mathtt{c}} = \{4\}$




Universe $\{1,2,3,4\}$

$$< \; = \; \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$$

## Trace Structures and Word Structures

The universe is an initial segment of natural numbers, i.e., $A = \{1, 2, \ldots, \ell\}$ and $<$ the usual order on them.

- finite trace structures are given by arbitrary unary relations $P_1$, $P_2$, ...
- word structures are given by unary relations $P_a$ for $a \in \Sigma$ *partitioning* the univserse

**Note.**

> Trace structures are in 1-1, onto, correspondence with word structures over $\{0,1\}^n$
>
> Word structures are in 1-1, onto, correspondence with words.

The last observation allows to consider whatever logic we invent for a word-structures from a point of view of formal languages: "What are the languages, i.e., sets of strings, that can be specified by this or that logic?"

## 2.2 Linear Temporal Logic

Even though, technically, the structures we are considering are first-order structures, we start by considering a language specially tailored to express the temporal aspect.

### Syntax of Linear Temporal Logic

The set $\mathsf{LTL}[p_1, \ldots, p_n]$ of *LTL-formulae* is freely generated

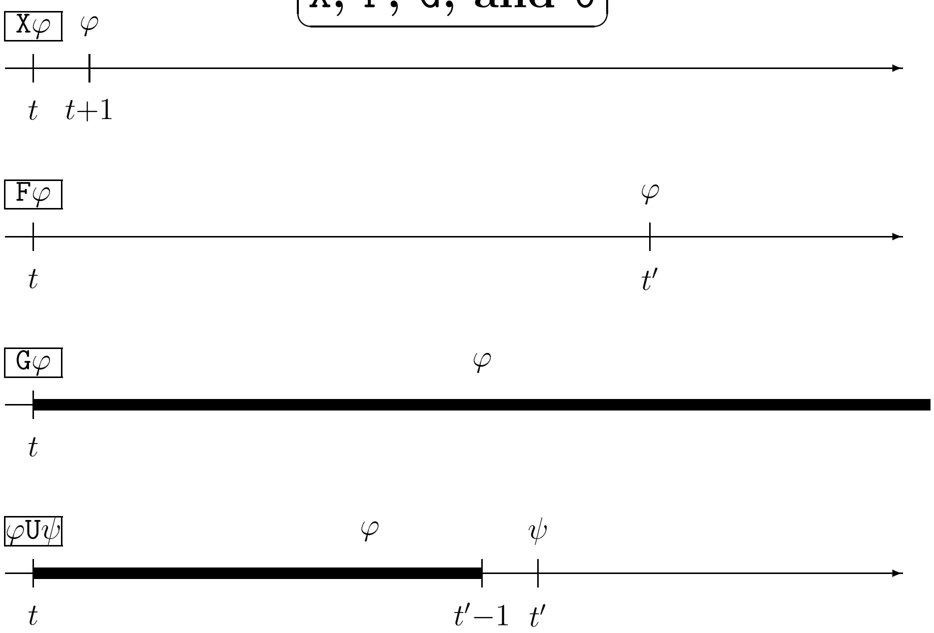
- $\top$, $\bot$, and all $p_i \in \{p_1, \ldots, p_n\}$
- If $\varphi, \psi \in \mathsf{LTL}[p_1, \ldots, p_n]$, then also $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$.
- ... and also
  - $\mathtt{X}\varphi$ "next"
  - $\mathtt{F}\varphi$ "finally"
  - $\mathtt{G}\varphi$ "globally"
  - $(\varphi\mathtt{U}\psi)$ "until"

Again, all the same comments about freely generated sets apply as for propositional logic. We just abbreviated this statement slightly on the slide. As usual, we omit unneeded parentheses when writing down LTL-formulae.

The formula $\mathtt{F}\varphi$ is often also pronounced as "eventually $\varphi$" and the formula $\mathtt{G}\varphi$ as "always $\varphi$".

Note that $\mathsf{LTL}[p_1, \ldots, p_n]$ properly extends the syntax of propositional logic.

## X, F, G, and U




## Semantics of LTL

Let $\mathcal{A} = (\{1, \ldots, \ell\}, <, P_1, P_2, \ldots, P_n)$ and $t \in \{1, \ldots, \ell\}$.
Define $\mathcal{A}, t \models \varphi$ for $\varphi \in \mathsf{LTL}[p_1, \ldots, p_2]$ inductively.

- $\mathcal{A}, t \models \top$; $\mathcal{A}, t \not\models \bot$; $\mathcal{A}, t \models p_i$ iff $t \in P_i$
- $\mathcal{A}, t \models \varphi \wedge \psi$ iff ...
- $\mathcal{A}, t \models \mathtt{X}\varphi$ iff $t < \ell$ and $\mathcal{A}, t{+}1 \models \varphi$
- $\mathcal{A}, t \models \mathtt{F}\varphi$ iff $\mathcal{A}, t' \models \varphi$ for some $t' \geq t$
- $\mathcal{A}, t \models \mathtt{G}\varphi$ iff $\mathcal{A}, t' \models \varphi$ for all $t' \geq t$
- $\mathcal{A}, t \models \varphi\mathtt{U}\psi$ iff there is some $t' \geq t$ s.t.
  $\mathcal{A}, t' \models \psi$ and $\mathcal{A}, t'' \models \varphi$ for all $t \leq t'' < t'$