CS-316, Logic and Semantics | **Coursework Assignment 1**
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1. Consider a mapping $\{1, 2, 3\} \to \{1, 2\}$. We want to reason about it using the variables $p_{11}, p_{12}, p_{21}, p_{22}, p_{31}, p_{32}$ where the meaning of p_{ij} is "i is mapped to j".

   - Write down a formula expressing that the mapping is a function. That is, write down formulae expressing
     - Every i is mapped to some j
     - Each i is mapped to at most one j
   - Write down a formula expressing that the mapping is onto.
   - Write down a formula expressing that the mapping is one-to-one.

2. Let $f\colon \{0, 1\}^3 \to \{0, 1\}$ given by the following table.

p_1	p_2	p_3	$f((p_1, p_2, p_3))$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

   Provide some formula $\varphi \in \mathsf{PL}[p_1, p_2, p_3]$ in disjunctive normal form representing φ.

3. In the lecture we have shown that every formula can be expressed using `nand` as only connective. Write down some formula equivalent to $p_1 \leftrightarrow p_2$ that uses only `nand`, and the variables p_1 and p_2. Prove that your formula has the desired property. (You may, e.g., use truth-tables to show that two formulae are equivalent.)

4. Show that every boolean function can be defined using (besides the projections) only `nor`, where

x	y	x `nor` y
0	0	1
0	1	0
1	0	0
1	1	0

5. Prove that for every $f\colon \{0,1\}^n \to \{0,1\}$ there is a formula φ of the shape

$$\bigwedge_j \bigvee_i \xi_{i,j}$$

where each ξ_{ij} is either x_i or $\neg x_i$, such that for all $\underline{a} \in \{0,1\}^n$ we have $f(\underline{a}) = 1$ if and only if $\underline{a} \models \varphi$.