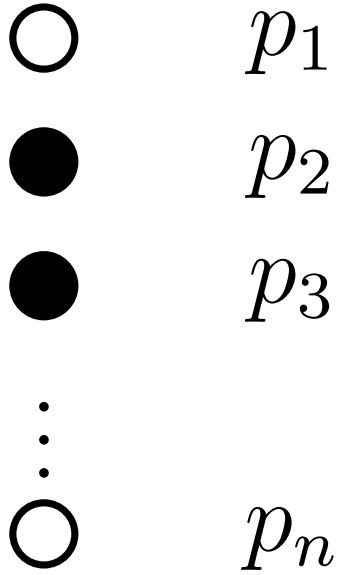
Propositional Logic



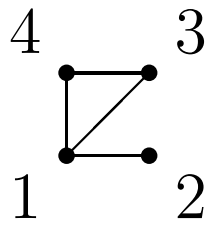
Syntax of Propositional Logic

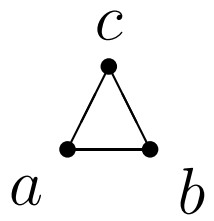
The set $\mathsf{PL}[p_1, \ldots, p_n]$ of *propositional formulae over* $p_1, \ldots, p_n$ is freely generated as follows.

- $\top$, $\bot$, and all $p_i \in \{p_1, \ldots, p_n\}$ are propositional formulae *(so called "atomic formulae").*
- If φ is a propositional formula, then so is $\neg\varphi$.
- If φ and ψ are propositional formulae, then so are $(\varphi \wedge \psi)$ and $(\varphi \vee \psi)$.

Example

Let $G = (\{1, 2, 3, 4\}, E)$ be an undirected graph. It can be described by formulae in $\mathsf{PL}[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}]$ where p_{ij} expresses the fact that there is an edge from i to j.

(a) Express: The graph is 

(b) Express: The graph contains some triangle. 

Evaluation of a Boolean Formula

For $\varphi \in \mathsf{PL}[p_1, \ldots, p_n]$ and $\underline{\mathrm{a}} = (a_1, \ldots, a_n) \in \{0,1\}^n$ we define $\varphi[\underline{\mathrm{a}}] \in \{0,1\}$ as follows.

- $\top[\underline{\mathrm{a}}] = 1$, $\bot[\underline{\mathrm{a}}] = 0$, $p_i[\underline{\mathrm{a}}] = a_i$
- $(\neg\varphi)[\underline{\mathrm{a}}] = \curlywedge(\varphi[\underline{\mathrm{a}}])$
- $(\varphi \wedge \psi)[\underline{\mathrm{a}}] = (\varphi[\underline{\mathrm{a}}]) \mathbb{\wedge} (\psi[\underline{\mathrm{a}}])$,
  $(\varphi \vee \psi)[\underline{\mathrm{a}}] = (\varphi[\underline{\mathrm{a}}]) \mathbb{\vee} (\psi[\underline{\mathrm{a}}])$

The functions $\curlywedge, \mathbb{\wedge}, \mathbb{\vee}$ are given by

x	y	$\curlywedge y$	$x \mathbb{\wedge} y$	$x \mathbb{\vee} y$
0	0	1	0	0
0	1	0	0	1
1	0		0	1
1	1		1	1

Model Relation for Propositional Logic

Writing $\underline{a} \models \varphi$ for $\varphi[\underline{a}] = 1$ we obtain the following.

- $\underline{a} \models \top$ always holds and $\underline{a} \models \bot$ never holds
- $\underline{a} \models \neg\varphi$ holds iff $\underline{a} \models \varphi$ does *not* hold
- $\underline{a} \models \varphi \wedge \psi$ holds if $\underline{a} \models \varphi$ *and* $\underline{a} \models \psi$ both hold.
  $\underline{a} \models \varphi \vee \psi$ holds if $\underline{a} \models \varphi$ holds *or* $\underline{a} \models \psi$ holds.

Expressive Completeness

Theorem. For every $f\colon \{0,1\}^n \to \{0,1\}$ there is some $\varphi \in \mathsf{PL}[p_1, \ldots, p_n]$ such that for all $\underline{\mathrm{a}} \in \{0,1\}^n$ we have $f(\underline{\mathrm{a}}) = \varphi[\underline{\mathrm{a}}]$.

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In fact, φ can be chosen to be of the form

$$\bigvee_j \bigwedge_i \xi_{ij} \qquad \text{with } \xi_{ij} \in \{x_i, \neg x_i\}$$

(in "disjunctive normal form")

Other Complete Sets of Connectives

- $\wedge$, $\neg$. *Indeed,* $x \mathbin{\mathbb{V}} y = \neg((\neg x) \mathbin{\mathbb{\wedge}} (\neg y))$.
- $\vee$, $\neg$
- `nand` where

x	y	x `nand` y
0	0	1
0	1	1
1	0	1
1	1	0

Indeed, $\neg x = x$ `nand` x *and* $x \mathbin{\mathbb{\wedge}} y = \neg(x$ `nand` $y)$.