The "semantic" proof of Büchi's Theorem provides a lot of insight into the nature of monadic second-order logic. However, it does not give us a way to construct the automaton we know it has to exist. Therefore it's worth knowing an alternative proof of Büchi's Theorem, which gives additional insights. We will sketch it here.

Alternative Proof

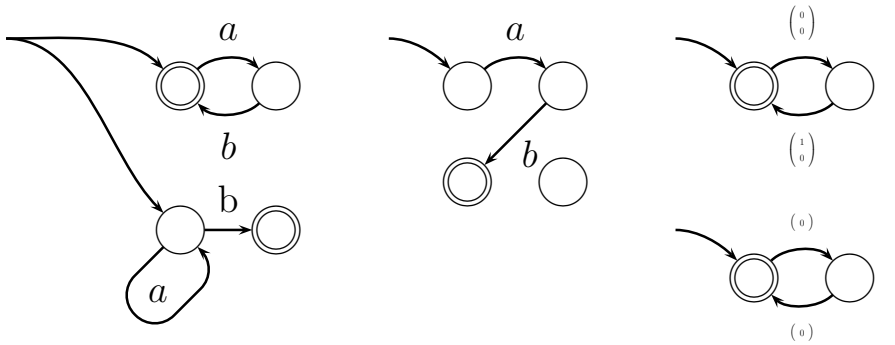
NFAs are closed under

- intersection $\mathcal{L} \cap \mathcal{L}'$
- union $\mathcal{L} \cup \mathcal{L}'$
- complement
  e.g., via power-set construction to get a DFA
- projection of the alphabet
  $\{\pi(a_1) \dots \pi(a_n) \mid a_1 \dots a_n \in \mathcal{L}\}$

of their languages.

Moreover, this closure is effective.

NFA Closure Properties



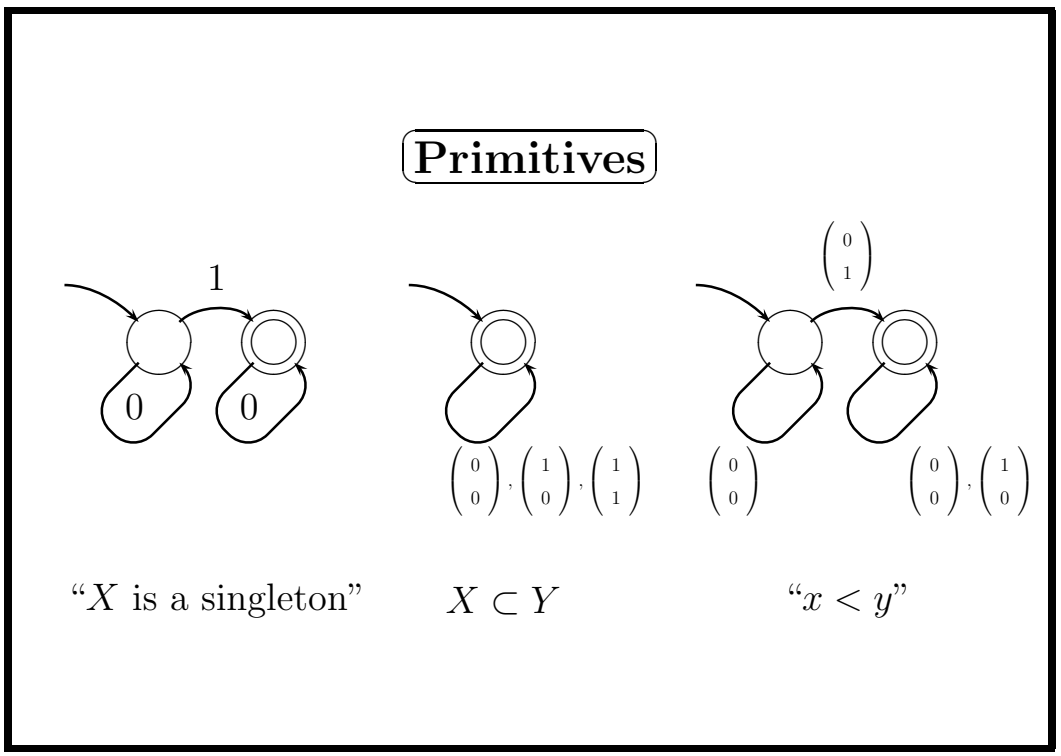
Once we have this observation, our strategy is, to first eliminate first-order(!!) quantification. $\exists x\varphi$ is replaced by

$$\exists X.(\text{“}X \text{ singleton”} \wedge \varphi[X/x])$$

The formula $x \in Y$ then becomes

$$X \subset Y$$

Note that both primitives (“singleton”, inclusion, order) can be easily defined by automata working on the corresponding trace structures.



Then the above observations suffice, as intersection, union, and complement correspond to the boolean connectives. Projection of the language is existential set quantification.

This new has additional consequences. Since the construction of the automaton is constructive, we can decide satisfiability. For a given formula just construct the automaton and decide non-emptiness.

Corollary. Satisfiability for MSO on words is decidable.

Another interesting consequence of this decidability result is the decidability of Presburger arithmetic, that is, the first-order arithmetic of the natural numbers with addition as only operation.

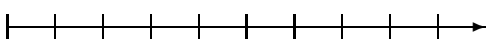
The idea is, that a predicate in a trace structure can be thought of as a natural number with the X corresponding to

$$\sum_{i \geq 1} 2^{i-1} \cdot X(i)$$

where the finiteness of the sum is guaranteed by the fact, that we consider only finite words.

Presburger Arithmetic

$X = 13$	1	0	1	1	0	0	0	0	0	0
$X = 30$	0	1	1	1	1	0	0	0	0	0
$Z = 43$	1	1	0	1	0	1	0	0	0	0



In this presentation, addition is MSO definable. Note that $X + Y = Z$ iff

$$\exists C.(\ (\forall i.\text{“}Z(i) = X(i) \oplus Y(i) \oplus C(i)\text{”}) \\ \wedge \forall i(C(i) \leftrightarrow \exists j(j \leq i \wedge X(j) \wedge Y(j) \wedge \forall k((j < k \wedge k < i) \rightarrow (X(k) \vee Y(k))))))$$

provided the universe is big enough, to hold the result, which can be expressed, e.g., by asserting $\forall i((X(i) \vee Y(i)) \rightarrow \exists j(i < j))$.

Corollary. Presburger Arithmetic is decidable.