MSO-Definability and Regular Language

For $m \geq 0$ we note that

- there are only finitely many $\equiv_m^{MSO}$ classes $[\![\mathcal{A}_w]\!]$, and
- the $\equiv_m^{MSO}$ class of wu only depends on that of w and u.

So

$$\begin{aligned} Q &= \{[\![\mathcal{A}_w]\!] \mid w \text{ a word}\} \\ \delta\colon\ Q \times \Sigma &\to Q \\ ([\![\mathcal{A}_w]\!], a) &\mapsto [\![\mathcal{A}_{wa}]\!] \end{aligned}$$

defines a finite automaton.

Büchi's Theorem

The following are equivalent for word languages $\mathcal{L} \subset \Sigma^+$.

- $\mathcal{L}$ is regular
- $\mathcal{L}$ is MSO definable, i.e., there is an MSO formula φ and

$$\mathcal{L} = \{w \mid w \models \varphi\}$$

Alternative Proof

NFAs are closed under

- intersection $\mathcal{L} \cap \mathcal{L}'$
- union $\mathcal{L} \cup \mathcal{L}'$
- complement
  e.g., via power-set construction to get a DFA
- projection of the alphabet
  $\{\pi(a_1) \ldots \pi(a_n) \mid a_1 \ldots a_n \in \mathcal{L}\}$

of their languages.

Moreover, this closure is effective.