The observation that $\equiv_m^{MSO}$ partitions, for every m, the class of all first-order structures into *finitely* many equivalence classes is the crucial tool towards obtaining an upper bound on the definable properties of monadic second-order logic. As usual, finitely many classes of indistinguishable objects leads to regular languages.

MSO-Definability and Regular Language

For $m \geq 0$ we note that

- there are only finitely many $\equiv_m^{MSO}$ classes $[\![\mathcal{A}_w]\!]$, and
- the $\equiv_m^{MSO}$ class of wu only depends on that of w and u.

So

$$\begin{aligned} Q &= \{[\![\mathcal{A}_w]\!] \mid w \text{ a word}\} \\ \delta\colon\ Q \times \Sigma &\to Q \\ ([\![\mathcal{A}_w]\!], a) &\mapsto [\![\mathcal{A}_{wa}]\!] \end{aligned}$$

defines a finite automaton.

Recall that the composition property can be easily seen by looking at games. If Duplicator has a strategy for m rounds on $w \mid w'$ and a strategy for m rounds on $u \mid u'$, he can combine them to a strategy for m rounds on $wu \mid w'u'$. Also note that this property is essential for the above "definition" to make sense!

Summing up, we get Büchi's Theorem.

Büchi's Theorem

The following are equivalent for word languages $\mathcal{L} \subset \Sigma^+$.

- $\mathcal{L}$ is regular
- $\mathcal{L}$ is MSO definable, i.e., there is an MSO formula φ and

$$\mathcal{L} = \{w \mid w \models \varphi\}$$

Proof. "$\Rightarrow$" Code up the run of an automaton
"$\Leftarrow$" Given φ, construct the automaton with $\equiv_m^{MSO}$ classes as states, where $m = \mathsf{qr}(\varphi)$.

Büchi's Theorem and its proofs have some interesting consequences.

> **Corollary.** There is $s\colon \mathbb{N} \to \mathbb{N}$ such that if φ has a word model, it has a word model of size at most $s(\mathsf{qr}(\varphi))$.

We know, by the pumping lemma, that if an automaton has an accepting run, then it has an accepting run on a word with at most as many letters as the automaton has states. But we used the same automaton for all the formulae with a given quantifier rank m.

> **Corollary.** If a of words is MSO-definable, then it is MSO-definable by a formula of the shape
> $$\exists X_1 \exists X_2 \ldots \exists X_n \varphi$$
> with φ first-order.

In other words, there is no need to alternate existential and universal quantification of second-order quantification. This is a completely different situation then in second-order arithmetic!

The reason is quite simple. If a property is MSO-definable, then the property, by Büchi's Theorem is regular. Hence there is a finite automaton defining it. But we can translate the property "$\mathfrak{A}$ has a successful run" into a formula of the given shape.

The "semantic" proof of Büchi's Theorem provides a lot of insight into the nature of monadic second-order logic. However, it does not give us a way to construct the automaton we know it has to exist. Therefore it's worth knowing an alternative proof of Büchi's Theorem, which gives additional insights. We will sketch it here.

Alternative Proof

NFAs are closed under

- intersection $\mathcal{L} \cap \mathcal{L}'$
- union $\mathcal{L} \cup \mathcal{L}'$
- complement
  e.g., via power-set construction to get a DFA
- projection of the alphabet
  $\{\pi(a_1) \ldots \pi(a_n) \mid a_1 \ldots a_n \in \mathcal{L}\}$

of their languages.

Moreover, this closure is effective.

Once we have this observation, our strategy is, to first eliminate first-order(!!) quantification. $\exists x \varphi$ is replaced by

$$\exists X.(\text{“}X \text{ singleton”} \wedge \varphi[X/x])$$

The formula $x \in Y$ then becomes

$$X \subset Y$$

Note that both primitives ("singleton", inclusion, order) can be easily defined by automata working on the corresponding trace structures.

Then the above observations suffice, as intersection, union, and complement correspond to the boolean connectives. Projection of the language is existential set quantification.

This new has additional consequences. Since the construction of the automaton is constructive, we can decide satisfiability. For a given formula just construct the automaton and decide non-emptiness.

> **Corollary.** Satisfiability for MSO on words is decidable.

Another interesting consequence of this decidability result is the decidability of Presburger arithmetic, that is, the first-order arithmetic of the natural numbers with addition as only operation.

The idea is, that a predicate in a trace structure can be thought of as a natural number with the X corresponding to

$$\sum_{i \geq 1} 2^{i-1} \cdot X(i)$$

where the finiteness of the sum is guaranteed by the fact, that we consider only finite words.

In this presentation, addition is MSO definable. Note that $X + Y = Z$ iff

$$\begin{aligned} \exists C.(\ & (\forall i.\text{“}Z(i) = X(i) \oplus Y(i) \oplus C(i)\text{”}) \\ & \wedge \forall i (C(i) \leftrightarrow \exists j (j \leq i \wedge X(j) \wedge Y(j) \wedge \forall k ((j < k \wedge k < i) \rightarrow (X(k) \vee Y(k)))))) \end{aligned}$$

provided the universe is big enough, to hold the result, which can be expressed, e.g., by asserting $\forall i ((X(i) \vee Y(i)) \rightarrow \exists j (i < j))$.

> **Corollary.** Presburger Arithmetic is decidable.