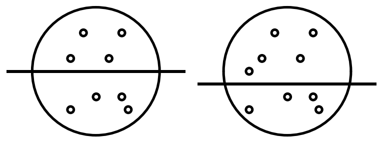
Now that we have developed a tool for obtaining bounds on the expressibility of Monadic Second Order Logic, let's use them and go back to the counting example. Remember, that first-order logic could count linearly far on unstructured sets and exponentially for on linear orders. MSO doesn't need an order to get exponentially far.

Example: Unstructured Sets

If $\mathcal{A} = (A)$ and $\mathcal{B} = (B)$ are structures over the empty signature, then

$$\mathcal{A} \equiv_m^{MSO} \mathcal{B} \text{ iff } (|A| = |B| \text{ or } |A|, |B| \geq 2^{m-1}).$$



Note that this observation in particular shows that, even though the word language $(\mathtt{aa})^+$ is MSO-definable, the property "The universe has even size" is not.

Proof. "$\Rightarrow$"
Let $|A| < |B|$ and $|A| < 2^{m-1}$. We describe a strategy for Spoiler to win in at most m moves.
The case $m = 1$ is trivial.
Let $m \geq 2$. If $|A| \leq 2^{m-1}-2$ Spoiler "splits" (using a set move) it into two small sets, i.e., of size $< 2^{m-2}$. No matter how Duplicator splits B, the induction hypothesis applies to one part. Otherwise, $|B| \geq 2^{m-1}$ and can be spilt into two big parts (of size at least 2^{m-2}). No matter how Duplicator splits A, one part will be small and the induction hypothesis applies.

"$\Leftarrow$" Let $|A|, |B| \geq 2^{m-1}$. We describe a strategy for Duplicator to play m rounds.
The case $m = 1$ is easy.
Let $m \geq 2$. Point moves of Spoiler are handled trivially; set-moves split one set, say A. Since $|A| \geq 2^{m-1}$, at most one part can be small. This part is then matched exactly by Duplicator.

The observation that $\equiv_m^{MSO}$ partitions, for every m, the class of all first-order structures into *finitely* many equivalence classes is the crucial tool towards obtaining an upper bound on the definable properties of monadic second-order logic. As usual, finitely many classes of indistinguishable objects leads to regular languages.

MSO-Definability and Regular Language

For $m \geq 0$ we note that

- there are only finitely many $\equiv_m^{MSO}$ classes $[\![\mathcal{A}_w]\!]$, and
- the $\equiv_m^{MSO}$ class of wu only depends on that of w and u.

So

$$\begin{aligned} Q &= \{[\![\mathcal{A}_w]\!] \mid w \text{ a word}\} \\ \delta\colon \quad Q \times \Sigma &\to Q \\ ([\![\mathcal{A}_w]\!], a) &\mapsto [\![\mathcal{A}_{wa}]\!] \end{aligned}$$

defines a finite automaton.

Recall that the composition property can be easily seen by looking at games. If Duplicator has a strategy for m rounds on $w \mid w'$ and a strategy for m rounds on $u \mid u'$, he can combine them to a strategy for m rounds on $wu \mid w'u'$. Also note that this property is essential for the above “definition” to make sense!

Summing up, we get Büchi's Theorem.

Büchi's Theorem

The following are equivalent for word languages $\mathcal{L} \subset \Sigma^+$.

- $\mathcal{L}$ is regular
- $\mathcal{L}$ is MSO definable, i.e., there is an MSO formula φ and

$$\mathcal{L} = \{w \mid w \models \varphi\}$$

Proof. "$\Rightarrow$" Code up the run of an automaton
"$\Leftarrow$" Given φ, construct the automaton with $\equiv_m^{MSO}$ classes as states, where $m = \mathsf{qr}(\varphi)$.

Büchi's Theorem and its proofs have some interesting consequences.

> **Corollary.** There is $s\colon \mathbb{N} \to \mathbb{N}$ such that if φ has a word model, it has a word model of size at most $s(\mathsf{qr}(\varphi))$.

We know, by the pumping lemma, that if an automaton has an accepting run, then it has an accepting run on a word with at most as many letters as the automaton has states. But we used the same automaton for all the formulae with a given quantifier rank m.

> **Corollary.** If a of words is MSO-definable, then it is MSO-definable by a formula of the shape
> $$\exists X_1 \exists X_2 \ldots \exists X_n \varphi$$
> with φ first-order.

In other words, there is no need to alternate existential and universal quantification of second-order quantification. This is a completely different situation then in second-order arithmetic!

The reason is quite simple. If a property is MSO-definable, then the property, by Büchi's Theorem is regular. Hence there is a finite automaton defining it. But we can translate the property "$\mathfrak{A}$ has a successful run" into a formula of the given shape.