MSO Games

$$\mathcal{A}, U_1, \ldots, U_k, a_1, \ldots, a_\ell \mid \mathcal{B}, V_1, \ldots, V_k, b_1, \ldots, b_\ell$$

where $U_1, \ldots, U_k \subset A$, $V_1, \ldots, V_k \subset B$.

Spoiler can choose between two types of moves.

- choose $a_{\ell+1} \in A$, or $b_{\ell+1} \in B$
- choose $U_{k+1} \subset A$, or $V_{k+1} \subset B$

Duplicator needs to keep the invariants

- $a_i = a_j$ iff $b_i = b_j$; $R_i^{\mathcal{A}}(a_{i_1}, \ldots, a_{i_\ell})$ iff $R_i^{\mathcal{B}}(b_{i_1}, \ldots, b_{i_\ell})$
- $a_i \in U_j$ iff $b_i \in V_j$

$\boxed{\textbf{MSO Games—The Theorem}}$

For any configuration $\mathcal{A}, \vec{U}, \vec{a} \mid \mathcal{B}, \vec{V}, \vec{b}$ in an MSO-game over finite structures, the following are equivalent.

- Duplicator can survive m more rounds.
- $\mathcal{A}, \vec{U}, \vec{a} \equiv_m^{MSO} \mathcal{B}, \vec{V}, \vec{b}$

Winning Condition for $\mathcal{A}, \vec{U}, \vec{a}$

Define $\chi_{m,\mathcal{A},\vec{U},\vec{a}}$ with $\mathsf{qr}(\chi_{m,\mathcal{A},\vec{U},\vec{a}}) \leq m$ s.t. $\mathcal{B}, \vec{V}, \vec{b} \models \chi_{m,\mathcal{A},\vec{U},\vec{a}}(\vec{x})$ iff Duplicator has a strategy for m rounds in $\mathcal{A}, \vec{U}, \vec{a} \mid \mathcal{B}, \vec{V}, \vec{b}$.

$$\begin{aligned}
\chi_{m+1,\mathcal{A},\vec{U},\vec{a}}(\vec{X}, \vec{x}) = \;& (\bigwedge_{a \in A} \exists y \chi_{m,\mathcal{A},\vec{U},\vec{a},a}) \wedge (\forall y \bigvee_{a \in A} \chi_{m,\mathcal{A},\vec{U},\vec{a},a}) \\
& \wedge (\bigwedge_{U \subset A} \exists Y \chi_{m,\mathcal{A},\vec{U},U,\vec{a}}) \wedge (\forall Y \bigvee_{U \subset A} \exists Y \chi_{m,\mathcal{A},\vec{U},U,\vec{a}})
\end{aligned}$$

$$\chi_{0,\mathcal{A},\vec{U},\vec{a}}(\vec{x}) = \bigwedge_{\substack{\mathcal{A}, \vec{a} \models \varphi \\ \varphi \text{ atomic}}} \varphi \quad \wedge \bigwedge_{\substack{\mathcal{A}, \vec{a} \not\models \varphi \\ \varphi \text{ atomic}}} \neg\varphi$$