We have seen that monadic second-order logic strictly extends first-order logic. What we don't have yet is an upper bound on how expressive monadic second-order is. For first-order logic, Ehrenfeucht-Fraïssé games turned out to be a useful tool. We now define a similar type of games.

MSO Games

$$\mathcal{A}, U_1, \ldots, U_k, a_1, \ldots, a_\ell \mid \mathcal{B}, V_1, \ldots, V_k, b_1, \ldots, b_\ell$$

where $U_1, \ldots, U_k \subset A$, $V_1, \ldots, V_k \subset B$.

Spoiler can choose between two types of moves.

- choose $a_{\ell+1} \in A$, or $b_{\ell+1} \in B$
- choose $U_{k+1} \subset A$, or $V_{k+1} \subset B$

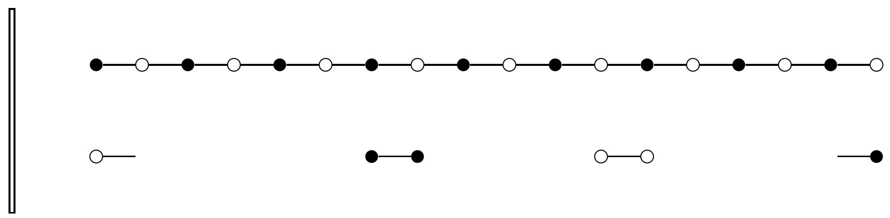
Duplicator needs to keep the invariants

- $a_i = a_j$ iff $b_i = b_j$; $R_i^{\mathcal{A}}(a_{i_1}, \ldots, a_{i_\ell})$ iff $R_i^{\mathcal{B}}(b_{i_1}, \ldots, b_{i_\ell})$
- $a_i \in U_j$ iff $b_i \in V_j$

In other words, the game is extended by the possibility of colouring a set of nodes.

Returning back to the example "even length", we immediately see that this extension of the game gives Spoiler a lot more power. He can now distinguish a linear order of odd length from one of even length by a fixed number of moves.

First, he colours every other node of the structure of even length.



No matter how Duplicator tries to match this colouring, Spoiler can either point out that the first or last node is coloured wrongly, or that there are two adjacent nodes of the same colour.

An analogous theorem as with first-order Ehrenfeucht-Fraïssé games holds.

MSO Games—The Theorem

For any configuration $\mathcal{A}, \vec{U}, \vec{a} \mid \mathcal{B}, \vec{V}, \vec{b}$ in an MSO-game over finite structures, the following are equivalent.

- Duplicator can survive m more rounds.
- $\mathcal{A}, \vec{U}, \vec{a} \equiv_m^{MSO} \mathcal{B}, \vec{V}, \vec{b}$

Here $\equiv_m^{MSO}$ denotes the relation of indistinguishability by MSO-formulae of quantifier rank up to m. The notion of quantifier rank is extended in a straight-forward way, by setting

$$\mathsf{qr}(\forall X \varphi) = \mathsf{qr}(\exists X \varphi) = 1 + \mathsf{qr}(\varphi)$$

The proof is quite analogous to the proof of the Ehrenfeucht-Fraïssé theorem.

Proof. "$\Rightarrow$"
Assume $\mathcal{A}, \vec{U}, \vec{a} \models \varphi$, $\mathcal{B}, \vec{V}, \vec{b} \not\models \varphi$, $\mathsf{qr}(\varphi) \leq m$.

Again, we have to describe a winning strategy for Spoiler. The only new case is handled by the additional move that Spoiler can do in an MSO-game.

Spoiler than can win in m rounds, using a strategy as in the EF-game. In the only new case $\varphi \equiv \exists X \psi$, there is some $U \subset A$ satisfying ψ and Spoiler will colour this set.

“$\Leftarrow$”

For the other direction we, again, just formalise the rules of the game.

Winning Condition for $\mathcal{A}, \vec{U}, \vec{a}$

Define $\chi_{m,\mathcal{A},\vec{U},\vec{a}}$ with $\mathsf{qr}(\chi_{m,\mathcal{A},\vec{U},\vec{a}}) \leq m$ s.t.
$\mathcal{B}, \vec{V}, \vec{b} \models \chi_{m,\mathcal{A},\vec{U},\vec{a}}(\vec{x})$ iff Duplicator has a strategy for m rounds in $\mathcal{A}, \vec{U}, \vec{a} \mid \mathcal{B}, \vec{V}, \vec{b}$.

$$\begin{aligned}
\chi_{m+1,\mathcal{A},\vec{U},\vec{a}}(\vec{X}, \vec{x}) = {} & \Big(\bigwedge_{a\in A} \exists y \chi_{m,\mathcal{A},\vec{U},\vec{a},a}\Big) \wedge \Big(\forall y \bigvee_{a\in A} \chi_{m,\mathcal{A},\vec{U},\vec{a},a}\Big) \\
& \wedge \Big(\bigwedge_{U\subset A} \exists Y \chi_{m,\mathcal{A},\vec{U},U,\vec{a}}\Big) \wedge \Big(\forall Y \bigvee_{U\subset A} \exists Y \chi_{m,\mathcal{A},\vec{U},U,\vec{a}}\Big)
\end{aligned}$$

$$\chi_{0,\mathcal{A},\vec{U},\vec{a}}(\vec{x}) = \bigwedge_{\substack{\mathcal{A},\vec{a} \models \varphi \\ \varphi \text{ atomic}}} \varphi \quad \wedge \bigwedge_{\substack{\mathcal{A},\vec{a} \not\models \varphi \\ \varphi \text{ atomic}}} \neg\varphi$$

Looking back to our formulae with bounded quantifier rank, we have seen that the set of structures are divided into equivalence classes by the relation $\equiv_m^{MSO}$, that can also be well described by appropriate games.

An important observation is that there are only finitely many of them.

Proposition. For every m and every finite $X \subset V \cup V^{(1)}$ there are, up to logical equivalence, only finitely many $\varphi \in \mathsf{MSO}[R_1, \ldots, R_n]$ with $\mathsf{qr}(\varphi) \leq m$ and $\mathsf{fv}(\varphi) \subset X$.

Proof. Induction on m. For $m = 0$ there are only finitely many atoms, hence only finitely many boolean combinations.
All φ with $\mathsf{qr}(\varphi) = m + 1$ are boolean combinations of formulae $\exists y\psi$ or $\forall y\psi$ with $\mathsf{qr}(\psi) = m$, of which there are only finitely many.