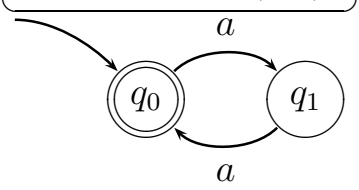
Example: $(aa)^*$



$$\begin{aligned}
\exists X_0 \exists X_1 [\ & \forall x((\forall y.x \leq y) \rightarrow (X_1(x) \wedge P_a(x))) \wedge \\
& \forall x \forall y(((x < y) \wedge \neg \exists z(x < z \wedge z < y)) \rightarrow \\
& \quad ((X_0(y) \rightarrow (X_1(x) \wedge P_a(y))) \wedge \\
& \quad (X_1(y) \rightarrow (X_0(x) \wedge P_a(y))))) \wedge \\
& \forall x((\forall y.y \leq x) \rightarrow X_0(y))]
\end{aligned}$$

Note that this way of describing a run one the one hand works with non-deterministic automata, and on the other hand enforces positive knowledge about the existence of a run. The given formula, for example, would not hold on the word `aabbaa`.

Also, as we have seen earlier, the property "even length" cannot be expressed in first-order logic. So, one consequence of the automaton we have just constructed is, the MSO strictly extends first-order logic.

Corollary. There are MSO-definable word-languages that are not first-order definable.