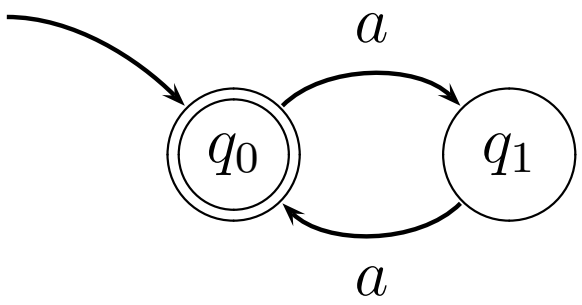
EF-Games over Linear Orders

Let $m \geq 1$ be a natural number, and $\mathcal{A} = (A, <^{\mathcal{A}})$ and $\mathcal{B} = (B, <^{\mathcal{B}})$ be linear orderings of lengths ℓ_A and ℓ_B, respectively.

Then $\mathcal{A} \equiv_m \mathcal{B}$ iff $(\ell_A = \ell_B$ or $\ell_A, \ell_B \geq 2^m - 1)$.

Parity of the Word Length



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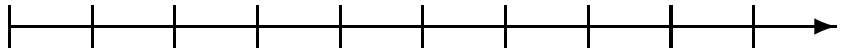

Syntax of Monadic Second-Order Logic

The set $\mathsf{MSO}[R_1, \ldots, R_n]$ of first-order formulae over the relation symbols $R_1, \ldots, R_n$ is freely generated as follows.

- $(x = y)$ for variables x, y
- $R_j x_{i_1} \ldots x_{i_{n_j}}$ if R_j is of arity n_j
- Xx for X predicate variable
- $\top$, $\bot$, $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$ for formulae ϕ, ψ
- $\forall x\varphi$ and $\exists x\varphi$ for φ a formula and x a variable
- $\forall X\varphi$ and $\exists X\varphi$ for φ a fla and X predicate var

Semantics of Monadic Second-Order Logic

Let $\mathcal{A} = (A, R_1^{\mathcal{A}}, \ldots)$ and $\eta\colon V \to A$, and $H\colon V^{(1)} \to \mathfrak{P}(A)$.
Define $\mathcal{A}, H, \eta \models \varphi$ for $\varphi \in \mathsf{MSO}[R_1, \ldots]$ inductively.

- $\mathcal{A}, H, \eta \models x = y$ iff $\eta(x) = \eta(y)$
  $\mathcal{A}, H, \eta \models R_1 y_1 \ldots y_n$ iff $(\eta(y_1), \ldots, \eta(y_n)) \in R_i^{\mathcal{A}}$
  $\boxed{\mathcal{A}, H, \eta \models Xx \text{ iff } \eta(x) \in H(X)}$
- $\mathcal{A}, H, \eta \models \varphi \wedge \psi$ iff $\ldots$
  $\mathcal{A}, H, \eta \models \forall x \varphi$ iff $\mathcal{A}, H, \eta_x^a \models \varphi$ for all $a \in A$
- $\boxed{\begin{array}{l}\mathcal{A}, H, \eta \models \forall X \varphi \text{ iff } \mathcal{A}, H_X^U, \eta \models \varphi \text{ for all } U \in \mathfrak{P}(A) \\ \mathcal{A}, H, \eta \models \exists X \varphi \text{ iff } \mathcal{A}, H_X^U, \eta \models \varphi \text{ for some } U \in \mathfrak{P}(A)\end{array}}$

Representing Automata Runs in MSO

Theorem. Let $\mathcal{L} \subset \Sigma^+$ be regular. Then there is an $\mathsf{MSO}[<, P_a, \ldots]$-formula φ such that

$$w \in \mathcal{L} \text{ iff } w \models \varphi$$

Run of an Automaton

Let $\mathfrak{A} = (Q, I, \Delta, F)$ be an NFA, $Q = \{q_0, \ldots, q_n\}$.

"$\mathfrak{A}$ has an accepting run": $\exists X_0 \ldots \exists X_n(\varphi_i \wedge \varphi_s \wedge \varphi_f)$

$$\varphi_i \equiv \forall x(\text{“}x \text{ first”} \rightarrow \bigwedge_j (X_j(x) \rightarrow \bigvee_{q_i \in I, (q_i,a,q_j) \in \Delta} P_a(x)))$$

$$\varphi_s \equiv \forall x \forall y(\chi_{\mathsf{next}}(x,y) \rightarrow \bigwedge_j (X_j(y) \rightarrow \bigvee_{(q_i,a,q_j \in \Delta)} (X_i(x) \wedge P_a(y))))$$

$$\varphi_f \equiv \forall x(\text{“}x \text{ last”} \rightarrow \bigvee_{q_j \in F} X_j(x))$$