Ehrenfeucht-Fraïssé Games

The game is played on configurations

$$\underbrace{(A, R_1^{\mathcal{A}}, R_2^{\mathcal{A}}, \ldots)}_{\mathcal{A}}, a_1, \ldots, a_k \mid \underbrace{(B, R_1^{\mathcal{B}}, R_2^{\mathcal{B}}, \ldots)}_{\mathcal{B}}, b_1, \ldots, b_k$$

where $a_1, \ldots, a_k \in A$ and $b_1, \ldots, b_k \in B$.

In each round, Spoiler picks either $a_{k+1} \in A$ or $b_{k+1} \in B$. Then Duplicator picks the other.

Duplicator needs to keep the invariants

- $a_i = a_j$ iff $b_i = b_j$
- $R_i^{\mathcal{A}}(a_{i_1}, \ldots, a_{i_\ell})$ iff $R_i^{\mathcal{B}}(b_{i_1}, \ldots, b_{i_\ell})$

Quantifier-Rank

Definition.

$\mathsf{qr}(\top) = \mathsf{qr}(\bot) = \mathsf{qr}(x = y) = \mathsf{qr}(Rx \ldots z) = 0$

$\mathsf{qr}(\neg\varphi) = \mathsf{qr}(\varphi)$

$\mathsf{qr}(\varphi \wedge \psi) = \mathsf{qr}(\varphi \vee \psi) = \max\{\mathsf{qr}(\varphi), \mathsf{qr}(\psi)\}$

$\mathsf{qr}(\forall x\varphi) = \mathsf{qr}(\exists x\varphi) = \mathsf{qr}(\varphi) + 1$

$\mathsf{FOL}_k[R_1, \ldots, R_n] = \{\varphi \in \mathsf{FOL}[R_1, \ldots, R_n] \mid \mathsf{qr}(\varphi) \leq k\}$

$\mathcal{A}, \vec{a} \equiv_k \mathcal{B}\vec{b}$ iff $\mathcal{A}, \vec{a} \equiv_{\mathsf{FOL}_k[\ldots]} \mathcal{B}, \vec{b}$

Ehrenfeucht-Fraïssé Theorem

Theorem. For any configuration $\mathcal{A}, \vec{a} \mid \mathcal{B}, \vec{b}$ in an EF-game over finite structures, the following are equivalent.

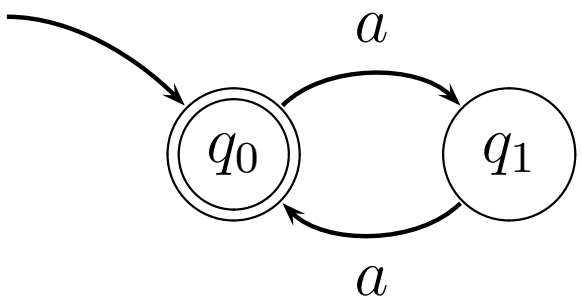
- Duplicator can survive m more rounds.
- $\mathcal{A}, \vec{a} \equiv_m \mathcal{B}, \vec{b}$

EF-Games over Linear Orders

Let $m \geq 1$ be a natural number, and $\mathcal{A} = (A, <^{\mathcal{A}})$ and $\mathcal{B} = (B, <^{\mathcal{B}})$ be linear orderings of lengths ℓ_A and ℓ_B, respectively.

Then $\mathcal{A} \equiv_m \mathcal{B}$ iff $(\ell_A = \ell_B$ or $\ell_A, \ell_B \geq 2^m - 1)$.

Parity of the Word Length



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Q_1 q_1 q_1 q_1 q_1 q_1

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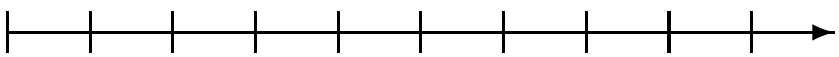

Syntax of Monadic Second-Order Logic

The set $\mathsf{MSO}[R_1, \ldots, R_n]$ of first-order formulae over the relation symbols $R_1, \ldots, R_n$ is freely generated as follows.

- $(x = y)$ for variables x, y
- $R_j x_{i_1} \ldots x_{i_{n_j}}$ if R_j is of arity n_j
- Xx for X predicate variable
- $\top$, $\bot$, $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$ for formulae ϕ, ψ
- $\forall x\varphi$ and $\exists x\varphi$ for φ a formula and x a variable
- $\forall X\varphi$ and $\exists X\varphi$ for φ a fla and X predicate var

Semantics of Monadic Second-Order Logic

Let $\mathcal{A} = (A, R_1^{\mathcal{A}}, \ldots)$ and $\eta\colon V \to A$, and $H\colon V^{(1)} \to \mathfrak{P}(A)$.
Define $\mathcal{A}, H, \eta \models \varphi$ for $\varphi \in \mathsf{MSO}[R_1, \ldots]$ inductively.

- $\mathcal{A}, H, \eta \models x = y$ iff $\eta(x) = \eta(y)$
  $\mathcal{A}, H, \eta \models R_1 y_1 \ldots y_n$ iff $(\eta(y_1), \ldots, \eta(y_n)) \in R_i^{\mathcal{A}}$
  $\boxed{\mathcal{A}, H, \eta \models Xx \text{ iff } \eta(x) \in H(X)}$
- $\mathcal{A}, H, \eta \models \varphi \wedge \psi$ iff $\ldots$
  $\mathcal{A}, H, \eta \models \forall x \varphi$ iff $\mathcal{A}, H, \eta_x^a \models \varphi$ for all $a \in A$
- $\boxed{\begin{array}{l}\mathcal{A}, H, \eta \models \forall X \varphi \text{ iff } \mathcal{A}, H_X^U, \eta \models \varphi \text{ for all } U \in \mathfrak{P}(A) \\ \mathcal{A}, H, \eta \models \exists X \varphi \text{ iff } \mathcal{A}, H_X^U, \eta \models \varphi \text{ for some } U \in \mathfrak{P}(A)\end{array}}$