Ehrenfeucht-Fraïssé Games

The game is played on configurations

$$\underbrace{(A, R_1^{\mathcal{A}}, R_2^{\mathcal{A}}, \ldots)}_{\mathcal{A}}, a_1, \ldots, a_k \;\mid\; \underbrace{(B, R_1^{\mathcal{B}}, R_2^{\mathcal{B}}, \ldots)}_{\mathcal{B}}, b_1, \ldots, b_k$$

where $a_1, \ldots, a_k \in A$ and $b_1, \ldots, b_k \in B$.

In each round, Spoiler picks either $a_{k+1} \in A$ or $b_{k+1} \in B$. Then Duplicator picks the other.

Duplicator needs to keep the invariants

- $a_i = a_j$ iff $b_i = b_j$
- $R_i^{\mathcal{A}}(a_{i_1}, \ldots, a_{i_\ell})$ iff $R_i^{\mathcal{B}}(b_{i_1}, \ldots, b_{i_\ell})$

Quantifier-Rank

Definition.

$\mathsf{qr}(\top) = \mathsf{qr}(\bot) = \mathsf{qr}(x = y) = \mathsf{qr}(Rx \ldots z) = 0$

$\mathsf{qr}(\neg\varphi) = \mathsf{qr}(\varphi)$

$\mathsf{qr}(\varphi \wedge \psi) = \mathsf{qr}(\varphi \vee \psi) = \max\{\mathsf{qr}(\varphi), \mathsf{qr}(\psi)\}$

$\mathsf{qr}(\forall x\varphi) = \mathsf{qr}(\exists x\varphi) = \mathsf{qr}(\varphi) + 1$

$\mathsf{FOL}_k[R_1, \ldots, R_n] = \{\varphi \in \mathsf{FOL}[R_1, \ldots, R_n] \mid \mathsf{qr}(\varphi) \leq k\}$

$\mathcal{A}, \vec{a} \equiv_k \mathcal{B}\vec{b}$ iff $\mathcal{A}, \vec{a} \equiv_{\mathsf{FOL}_k[\ldots]} \mathcal{B}, \vec{b}$

Ehrenfeucht-Fraïssé Theorem

Theorem. For any configuration $\mathcal{A}, \vec{a} \mid \mathcal{B}, \vec{b}$ in an EF-game over finite structures, the following are equivalent.

- Duplicator can survive m more rounds.
- $\mathcal{A}, \vec{a} \equiv_m \mathcal{B}, \vec{b}$

Winning Condition for $\mathcal{A}, \vec{a}$

Define $\chi_{m,\mathcal{A},\vec{a}}$ with $\mathsf{qr}(\chi_{m,\mathcal{A},\vec{a}}) \leq m$ as.t. $\mathcal{B}, \vec{b} \models \chi_{m,\mathcal{A},\vec{a}}(\vec{x})$ iff Duplicator has a strategy for m round in $\mathcal{A}, \vec{a} \mid \mathcal{B}, \vec{b}$.

$$\chi_{m+1,\mathcal{A},\vec{a}}(\vec{x}) = (\bigwedge_{a \in A} \exists y \chi_{m,\mathcal{A},\vec{a},a}) \wedge (\forall y \bigvee_{a \in A} \chi_{m,\mathcal{A},\vec{a},a})$$

$$\chi_{0,\mathcal{A},\vec{a}}(\vec{x}) = \bigwedge_{\substack{\mathcal{A}, \vec{a} \models \varphi \\ \varphi \text{ atomic}}} \varphi \quad \wedge \quad \bigwedge_{\substack{\mathcal{A}, \vec{a} \not\models \varphi \\ \varphi \text{ atomic}}} \neg\varphi$$

Example

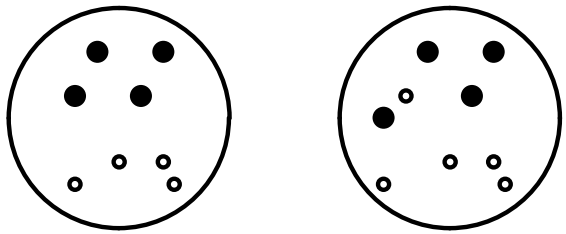
Let $\mathcal{A} = (\{1, 2, 3\}, <, P_a, P_b)$ be the structure for the word $\mathtt{aab}$, i.e. $P_a = \{1, 2\}, P_b = \{3\}$.

- Write down $\chi_{1,\mathcal{A},1}$.
- Does $\mathtt{aaab}, 1 \models \chi_{1,\mathcal{A},1}$?

Example: Unstructured Sets

If $\mathcal{A} = (A)$ and $\mathcal{B} = (B)$ are structures over the empty signature, then

$$\mathcal{A} \equiv_m \mathcal{B} \text{ iff } (|A| = |B| \text{ or } |A|, |B| \geq m).$$



EF-Games over Linear Orders

Let $m \geq 1$ be a natural number, and $\mathcal{A} = (A, <^{\mathcal{A}})$ and $\mathcal{B} = (B, <^{\mathcal{B}})$ be linear orderings of lengths ℓ_A and ℓ_B, respectively.

Then $\mathcal{A} \equiv_m \mathcal{B}$ iff $(\ell_A = \ell_B$ or $\ell_A, \ell_B \geq 2^m - 1)$.