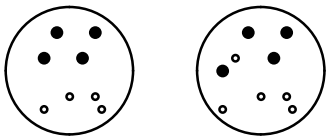
Example: Unstructured Sets

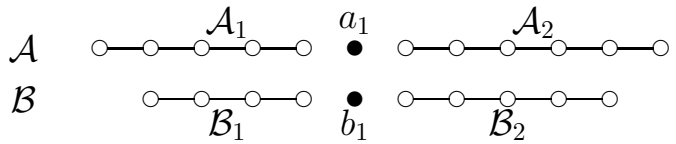
If $\mathcal{A} = (A)$ and $\mathcal{B} = (B)$ are structures over the empty signature, then

$$\mathcal{A} \equiv_m \mathcal{B} \text{ iff } (|A| = |B| \text{ or } |A|, |B| \geq m).$$



We now study Ehrenfeucht-Fraïssé games over linear order, that is, words over the one letter alphabet. Our results will, in particular, imply that the (regular!) property "the length is even" is *not* first-order definable.

The crucial observation is that a pair of pebbles completely separates the game into two independent ones.



For every move of Spoiler in one part, Duplicator has to play in the same part to avoid losing immediately due to order. But, if he loses following this strategy, he has also lost one of the part games, as all the order can say is, that one part game is to the one side of the initial pebble pair and the other part game to the other side. So, the best Spoiler can do is to chose one of the part games and try and win there.

If $\mathcal{A}_1 \equiv_m \mathcal{B}_1$ and $\mathcal{A}_2 \equiv_m \mathcal{B}_2$ then
$\mathcal{A}_1 \oplus \{a_1\} \oplus \mathcal{A}_2 \equiv_m \mathcal{B}_1 \oplus \{b_1\} \oplus \mathcal{B}_2$.

And vice versa!